Lecture No - 9   Examples

First of all we revise the example which we did in our 8^{th} lecture.
Consider w = f(x,y,z) Where
x = g(t), y = f(t), z = h(t)
Then

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

Example:

$$\mathbf{w = x^2 + y + z + 4}$$
$$\mathbf{x = e^t, \quad y = \cos t, \quad z = t + 4}$$

$$\frac{\partial w}{\partial x} = 2x, \quad \frac{\partial w}{\partial y} = 1, \quad \frac{\partial w}{\partial z} = 1$$

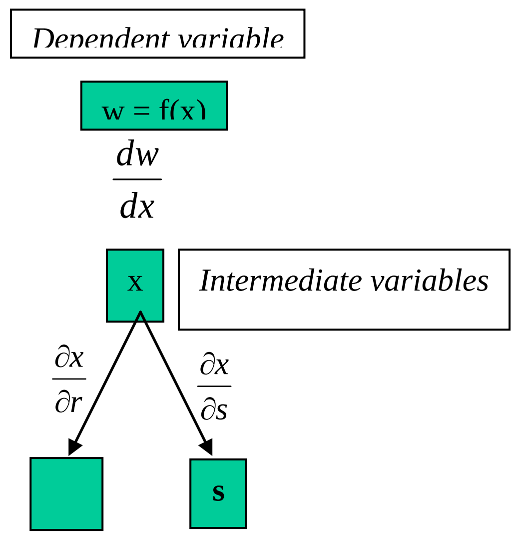
$$\frac{dx}{dt} = e^t, \quad \frac{dy}{dt} = -\text{Sin}t, \quad \frac{dz}{dt} = 1$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$$
$$= (2x)(e^t) + (1).(-\text{Sin}t) + (1)(1)$$
$$= 2(e^t)(e^t) - \text{Sin}t + 1$$
$$= 2e^{2t} - \text{Sin}t + 1$$

Consider

w = f(x), where x = g(r, s). Now it is clear from the figure that “x” is intermediate variable and we can write.

$$\frac{\partial w}{\partial r} = \frac{dw}{dx}\frac{\partial x}{\partial r} \text{ and } \frac{\partial w}{\partial s} = \frac{dw}{dx}\frac{\partial x}{\partial s}$$



Example:

$$\mathbf{w = \text{Sin}\,x + x^2, \quad x = 3r + 4s}$$

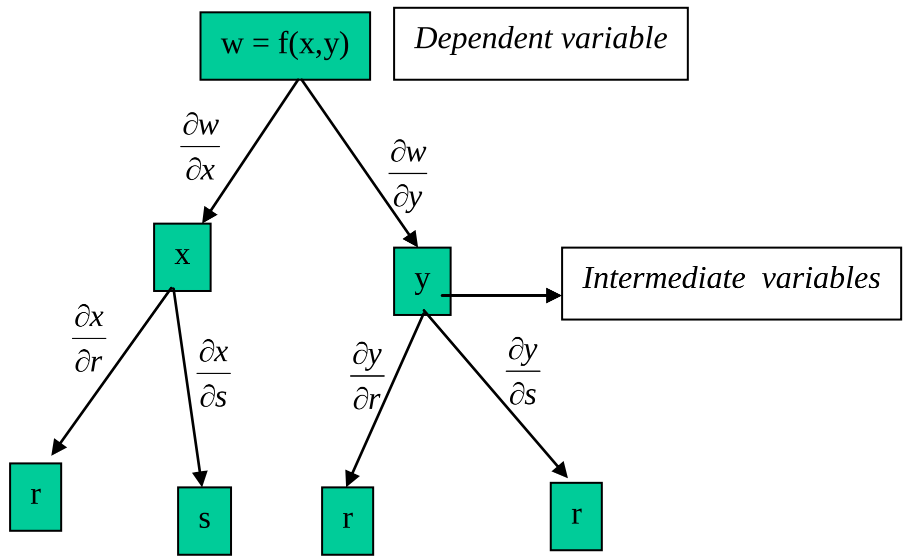
$$\frac{dw}{dx} = \text{Cos}x + 2x$$

$$\frac{\partial x}{\partial r} = 3 \qquad \frac{\partial x}{\partial s} = 4$$

$$\frac{\partial w}{\partial r} = \frac{dw}{dx}\cdot\frac{\partial x}{\partial r}$$
$$= (\text{Cos}x + 2x) . 3$$
$$= 3\,\text{Cos}\,(3r+4s) + 6\,(3r + 4s)$$
$$= 3\,\text{Cos}\,(3r + 4s) + 18r + 24s$$

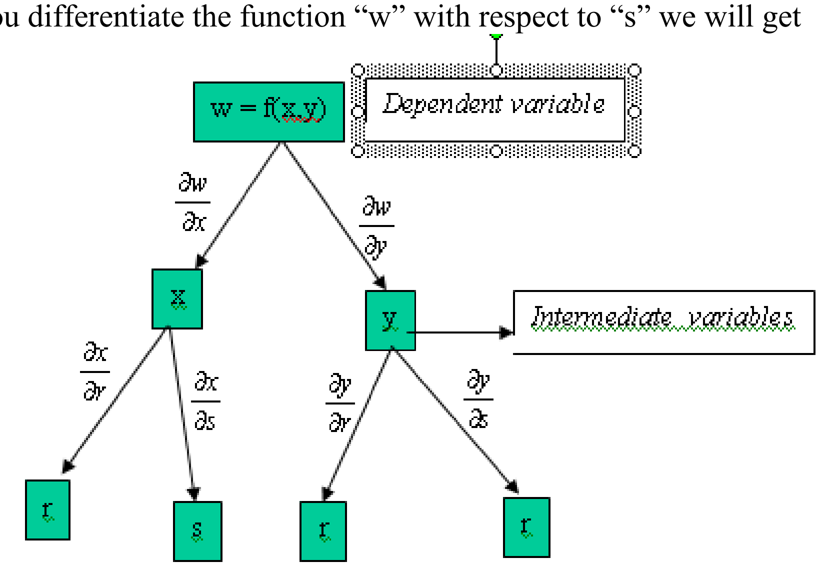
$$\frac{\partial w}{\partial s} = \frac{dw}{dx}\cdot\frac{\partial x}{\partial s}$$
$$= (\text{Cos}x + 2x) . 4$$
$$= 4\,\text{Cos}x + 8x$$
$$= 4\,\text{Cos}\,(3r + 4s) + 8\,(3r + 4s)$$
$$= 4\,\text{Cos}\,(3r + 4s) + 24r + 32s$$

Consider the function $w = f(x,y)$, *Where* $x = g(r, s), y = h(r, s)$



$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}$$

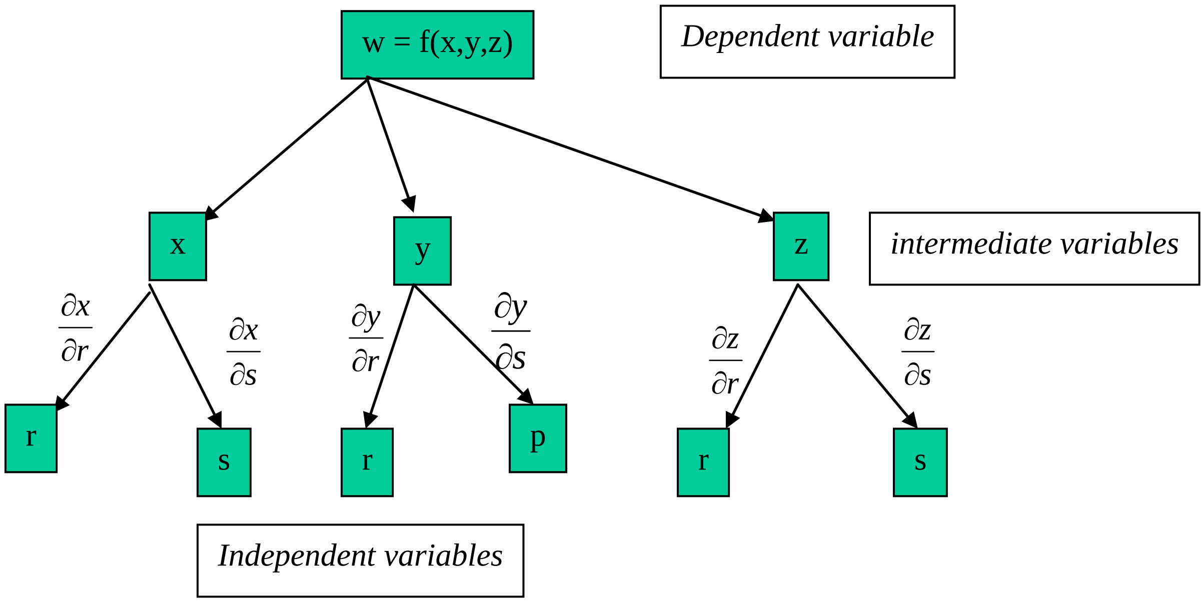
Similarly if you differentiate the function "w" with respect to "s" we will get



And we have

$$\frac{\partial w}{\partial s}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

Consider the function *w = f(x,y,z), Where x = g(r, s), y = h(r,s), z = k(r, s)*



Thus we have

$$\frac{\partial w}{\partial r}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial r}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

Similarly if we differentiate with respect to "s" then we have,

$$\frac{\partial w}{\partial s}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial s}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

Example:

Consider the function $w=x+2y+z^2,\ x=\frac{r}{s},\ y=r^2+\ln s,\ z=2r$

First we will calculate

$$\frac{\partial w}{\partial x}=1\quad \frac{\partial w}{\partial y}=2\quad \frac{\partial w}{\partial z}=2z\quad \frac{\partial x}{\partial r}=\frac{1}{s}\quad \frac{\partial y}{\partial r}=2r\quad \frac{\partial z}{\partial r}=2$$

Now as we know that $\frac{\partial w}{\partial r}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial r}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$ By putting the values from above we get

$$\frac{\delta w}{\delta r}=(1)\left(\frac{1}{s}\right)+(2)(2r)+(2z)(2)$$

$$=\frac{1}{s}+4r+(4r)(2)\ =\frac{1}{s}+12r$$

Now

$$\frac{\partial x}{\partial s} = -\frac{r}{s^2} \quad \frac{\partial y}{\partial s} = \frac{1}{s} \quad \frac{\partial z}{\partial s} = 0$$

So we can calculate

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$= (1)\left(-\frac{r}{s^2}\right) + (2)\left(\frac{1}{s}\right) + (2z)(0)$$

$$= \frac{2}{s} - \frac{r}{s^2}$$

Remembering the different Forms of the chain rule:

The best thing to do is to draw appropriate tree diagram by placing the dependent variable on top, the intermediate variables in the middle, and the selected independent variable at the bottom. To find the derivative of dependent variable with respect to the selected independent variable, start at the dependent variable and read down each branch of the tree to the independent variable, calculating and multiplying the derivatives along the branch. Then add the products you found for the different branches.

The Chain Rule for Functions of Many Variables

Suppose $\omega = f(x, y, \ldots, \upsilon)$ is a differentiable function of the variables x, y, ….., υ (a finite set) and the x, y, …, υ are differentiable functions of p, q, , t (another finite set). Then ω is a differentiable function of the variables p through t and the partial derivatives of ω with respect to these variables are given by equations of the form

$$\frac{\partial \omega}{\partial p} = \frac{\partial \omega}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial \omega}{\partial y}\frac{\partial y}{\partial p} + \ldots\ldots + \frac{\partial \omega}{\partial \upsilon}\frac{\partial \upsilon}{\partial p}.$$

The other equations are obtained by replacing p by q, …, t, one at a time. One way to remember last equation is to think of the right-hand side as the dot product of two vectors with components.

$$\left(\frac{\partial\omega}{\partial x},\frac{\partial\omega}{\partial y}\cdots\cdots\frac{\partial\omega}{\partial \upsilon}\right) \quad \text{and} \quad \left(\frac{\partial x}{\partial p},\frac{\partial y}{\partial p}\cdots\cdots\frac{\partial \upsilon}{\partial p}\right)$$

Derivatives of ω with respect to the intermedaite variables

Derivatives of the intermedaite variables with respect to the selected independent variable

Example:

$$w = \ln(e^r + e^s + e^t + e^u)$$

Taking "ln" of both sides of the given equation we get

$$e^w = e^r + e^s + e^t + e^u$$

Now Taking partial derivative with respect to "r, s , u , and t" we get

$$e^w w_r = e^r \Rightarrow w_r = e^{r-w} \quad , \quad e^w w_s = e^s \Rightarrow w_s = e^{s-w} \quad , \quad e^w w_u = e^u \Rightarrow w_u = e^{u-w} \text{ and}$$

$$e^w w_t = e^t \Rightarrow w_t = e^{t-w}$$

Now since we have $w_r = e^{r-w}$ Now Differentiate it partially w.r.t. "s"

$$w_{rs} = e^{r-w}(-w_s)$$

$$= -e^{r-w}e^{s-w} \qquad \text{(Here we use the value of } w_s \text{)}$$

$$w_{rs} = -e^{r+s-2w}$$

Now differentiate it partially w.r.t. "t" and using the value of w_t we get,

$$w_{rst} = -e^{r+s-2w}(-2w_t)$$
$$= 2e^{r+s-2w}e^{t-w}$$
$$w_{rst} = 2e^{r+s+t-3w}$$

Now differentiate it partially w.r.t. "u" we get,

$w_{rstu} = 2e^{r+s-3w}(-3w_u)$ and by putting the value of w_u, we get,

$$w_{rstu} = -6e^{r+s+t-3w}(e^{u-w})$$

$$w_{rstu} = -6e^{r+s+t+u-4w}$$