Lecture No- 8 More About Euler Theorem Chain Rule

In general, the order of differentiation in an nth order partial derivatice can be change without affecting the final result whenever the function and all its partial derivatives of order ≤ n are continuous. For example, if f and its partial order derivatives of the first, second, and third orders are continuous on an open set, then at each point of the set,

$$f_{xyy} = f_{yxy} = f_{yyx}$$

or in another notation.

$$\frac{\partial^3 f}{\partial y^2 \partial x} = \frac{\partial^3 f}{\partial y \partial x \partial y} = \frac{\partial^3 f}{\partial x \partial y^2}$$

Order of differentiation

For a function

$$f(x,y) = y^2 x^4 e^x + 2$$

If we are interested to find $\frac{\partial^5 f}{\partial y^3 \partial x^2}$, that is, differentiating in the order firstly w.r.t. x and then w.r.t. y, calculation will involve many steps making the job difficult. But if we differentiate this function with respect to y, firstly and then with respect to x secondly then the value of this fifth order derivative can be calculated in a few steps.

$$\frac{\partial^5 f}{\partial x^2 \partial y^3} = 0$$

EXAMPLE

$$f(x,y) = \frac{x+y}{x-y}$$

$$f_x(x,y) = \frac{(x-y)\frac{\partial}{\partial x}(x+y) - (x+y)\frac{\partial}{\partial x}(x-y)}{(x-y)^2}$$

$$= \frac{(x-y)(1)-(x+y)(1)}{(x-y)^2}$$

$$= \frac{-2y}{(x-y)^2}$$

$$f_y(x,y) = \frac{(x-y)\frac{\partial}{\partial y}(x+y) - (x+y)\frac{\partial}{\partial y}(x-y)}{(x-y)^2}$$

$$= \frac{(x-y)(1)-(x+y)(-1)}{(x-y)^2}$$

$$= \frac{2x}{(x-y)^2}$$

EXAMPLE

$$f(x, y) = x^3 e^{-y} + y^3 \sec\sqrt{x}$$

$$f_x(x, y) = 3x^2 e^{-y} + y^3 \sec\sqrt{x}\tan\sqrt{x}\,\frac{1}{2\sqrt{x}}$$

$$f_y(x, y) = -x^3 e^{-y} + 3y^2 \sec\sqrt{x}$$

EXAMPLE

$$f(x, y) = x^2 y e^{xy}$$

$$\begin{aligned} f_x(x, y) &= 2xye^{xy} + x^2 y^2 e^{xy} \\ &= xye^{xy}(2 + xy) \\ f_x(1,1) &= (1)(1)e^{(1)(1)}[2 + (1)(1)] \\ &= 3e \end{aligned}$$

$$f(x, y) = x^2 y e^{xy}$$

$$\begin{aligned} f_y(x, y) &= x^2 e^{xy} + x^3 y e^{xy} \\ &= x^2 e^{xy}(1 + xy) \\ f_y(1,1) &= (1)(1)e^{(1)(1)}[1 + (1)(1)] \\ &= 2e \end{aligned}$$

Example

$$f(x, y) = x^2 \cos(xy)$$

$$\begin{aligned} f_x(x, y) &= 2x\cos(xy) - x^2 y \sin(xy) \\ f_x(1/2, \pi) &= 2(1/2)\cos(\pi/2) - (1/2)^2(\pi)\sin(\pi/2) \\ &= -\pi/4 \end{aligned}$$

$$f_y(x,y) = -x^3 \sin(xy)$$
$$f_y(\tfrac{1}{2},\pi) = -(\tfrac{1}{2})^3 \sin(\tfrac{\pi}{2})$$
$$= -\tfrac{1}{8}$$

EXAMPLE

$$w = (4x - 3y + 2z)^5$$
$$\frac{\partial w}{\partial x} = 20(4x-3y+2z)^4$$
$$\frac{\partial^2 w}{\partial y \partial x} = -24 \ (4x-3y+2z)^3$$
$$\frac{\partial^3 w}{\partial z \partial y \partial x} = -1440(4x-3y+2z)^2$$
$$\frac{\partial^4 w}{\partial z^2 \partial y \partial x} = -576 \ (4x-3y+2z)$$

Chain Rule in function of One variable

Given that w= f(x) and x = g(t), we find $\frac{dw}{dt}$ *as follows:*

From w = f(x), we get $\frac{dw}{dx}$

From x = g(t), we get $\frac{dx}{dt}$

Then

$$\frac{dw}{dt} = \frac{dw}{dx} \frac{dx}{dt}$$

Example

w = x + 4, x = Sint

By Substitution

w = Sint + 4

$$\frac{dw}{dt} = \text{Cost}$$

w = x + 4 $\Rightarrow \frac{dw}{dx} = 1$

x = Sint $\Rightarrow \frac{dx}{dt} = \text{Cost}$

By Chain Rule

$$\frac{dw}{dt} = \frac{dw}{dt} \times \frac{dx}{dt} = 1 \,.\times \text{Cost} = \text{Cost}$$

Chain rule in function of one variable

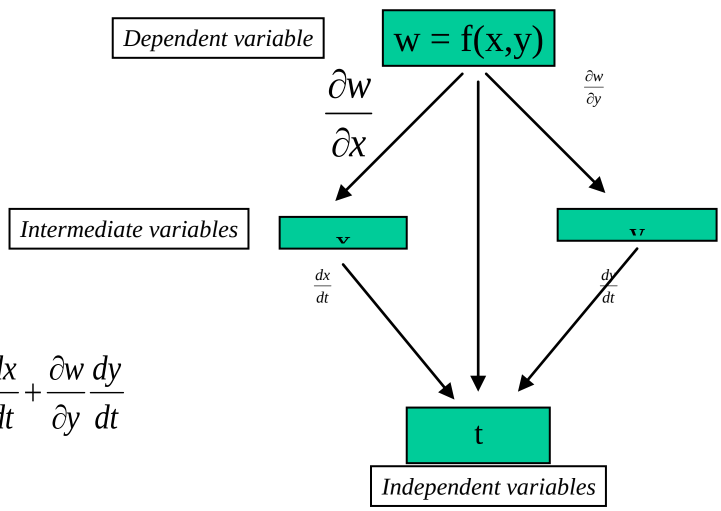
y is a function of u, u is a function of v
v is a function of w, w is a function of z
z is a function of x. Ultimately y is function of x

so we can talk about $\frac{dy}{dx}$

and by chain rule it is given by

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dw}\frac{dw}{dz}\frac{dz}{dx}$$

$w = f(x,y), \;\; x = g(t), y = f(t)$



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

<u>EXAMPLE BY SUBSTITUTION</u>

$$w = xy$$
$$x = \cos t, \;\; y = \sin t$$
$$w = \cos t \sin t$$
$$= \frac{1}{2} \; 2 \sin t \; \mathrm{cot}$$
$$= \frac{1}{2} \; \sin 2t$$
$$\frac{dw}{dt} = \frac{1}{2} \; \cos 2t.2$$
$$= \cos 2t$$

EXAMPLE

$$w = xy, x = \cos t, \text{and } y = \sin t$$

$$\frac{\partial w}{\partial y} = x \qquad \frac{\partial w}{\partial x} = y$$

$$\frac{dx}{dt} = -\sin t, \quad \frac{dy}{dt} = \cos t,$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

$$= (\sin t)(-\sin t) + (\cos t)(\cos t)$$

$$= -\sin^2 t + \cos^2 t = \cos 2t$$

EXAMPLE

$$\mathbf{z = 3x^2 y^3}$$

$$\mathbf{x = t^4 \;,\; y = t^2}$$

$$\frac{\partial z}{\partial x} = 6xy^3, \qquad \frac{\partial z}{\partial y} = 9x^2y^2$$

$$\frac{dx}{dt} = 4t^3, \qquad \frac{dy}{dt} = 2t$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (6xy^3)(4t^3) + 9x^2y^2(2t)$$

$$= 6(t^4)(t^6)(4t^3) + 9(t^8)(t^4)(2t)$$

$$= 24t^{13} + 18t^{13} = 42t^{13}$$

EXAMPLE

$$\mathbf{z = \sqrt{1 + x - 2xy^4}}$$

$$\mathbf{x = ln}$$

$$\mathbf{y = t}$$

$$\frac{\partial z}{\partial x} = \frac{1}{2\sqrt{1 + x - 2xy^4}}$$

$$\frac{\partial z}{\partial y} = \frac{1}{2\sqrt{1 + x - 2xy^4}} .(-8xy^3) = \frac{-4xy^3}{\sqrt{1 + x - 2xy^4}}$$

$$\frac{dx}{dt} = \frac{1}{t}, \qquad \frac{d}{dt} =$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$= \frac{1-2y^4}{2\sqrt{1+x-2xy^4}\ t}\ \frac{1}{t} - \frac{4xy^3}{\sqrt{1+x-2xy^4}}\cdot 1$$

$$= \frac{1}{\sqrt{1+x-2xy^4}}\left[\frac{1-}{2t} - 4xy^3\right]$$

$$= \frac{1}{\sqrt{1+\text{lint}-2(\ln t)\,t^4}}\left[\frac{1-2t^4}{2t} - 4(\ln t)\,t^3\right]$$

$$= \frac{1}{\sqrt{1+\ln t-2t^4\ln t}}\left[\frac{1}{2t} - t^3 - 4t^3\ln t\right]$$

EXAMPLE

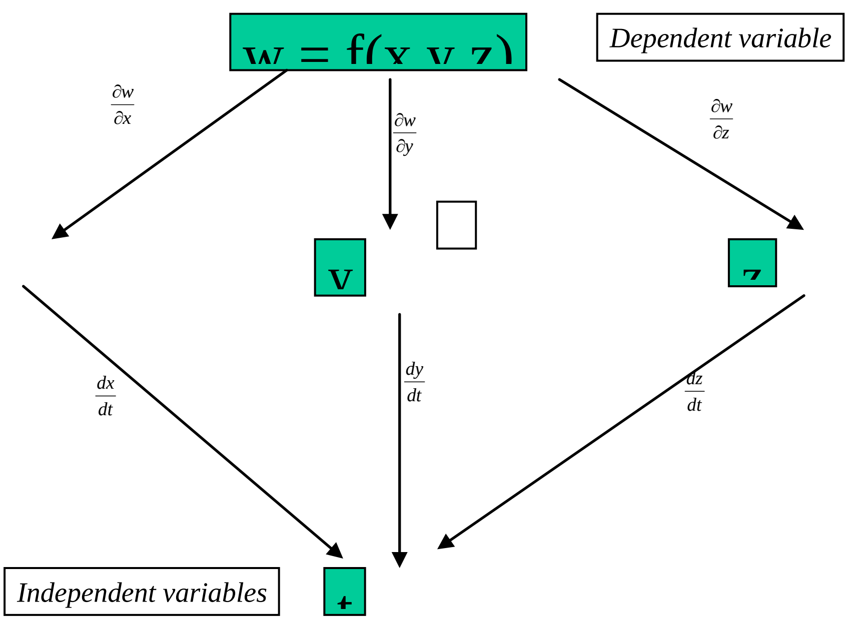
$$\mathbf{z = \ln(2x^2 + y)}$$

$$\mathbf{x = \sqrt{t}, \qquad y = t^{2/3}}$$

$$\frac{\partial z}{\partial x} = \frac{1}{2x^2+y}\cdot 4x = \frac{4}{2x^2+y}$$

$$\frac{\partial z}{\partial y} = \frac{1}{2x^2+y}, \quad \frac{dx}{dt} = \frac{1}{2}\frac{1}{\sqrt{t}}, \qquad \frac{dy}{dt} = \frac{2}{3}t^{-1/3}$$

w = f(x,y,z), x = g(t) ,y = f(t), z = h(t)



Overview of Lecture#8

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