Lecture No -7 Geometric meaning of partial derivative

<u>Geometric meaning of partial derivative</u>

$$\mathbf{z} = \boldsymbol{f(x, y)}$$

Partial derivative of f with respect of x is denoted by

$$\frac{\partial z}{\partial x} \text{ or } fx \text{ or } \frac{\partial f}{\partial x}$$

Partial derivative of f with respect of y is denoted by

$$\frac{\partial z}{\partial y} \text{ or } fy \text{ or } \frac{\partial f}{\partial y}$$

<u>Partial Derivatives</u>

Let z = f(x, y) be a function of two variable defined on a certain domain D.
For a given change Δx in x, keeping y as it is, the change Δz in z, is given by

$$\Delta \mathbf{z} = \mathbf{f}\,(\mathbf{x} + \Delta \mathbf{x}, \mathbf{y}) - \mathbf{f}\,(\mathbf{x}, \mathbf{y})$$

If the ratio

$$\frac{\Delta z}{\Delta x} = \frac{f(x + \Delta x,\ y) - f(x, y)}{\Delta x}$$

approaches to a finite limit as $\Delta x \to 0$, then this limit is called Partial derivative of f with respect of x.
Similarly for a given change Δy in y, keeping x as it is, the change Δz in z, is given by

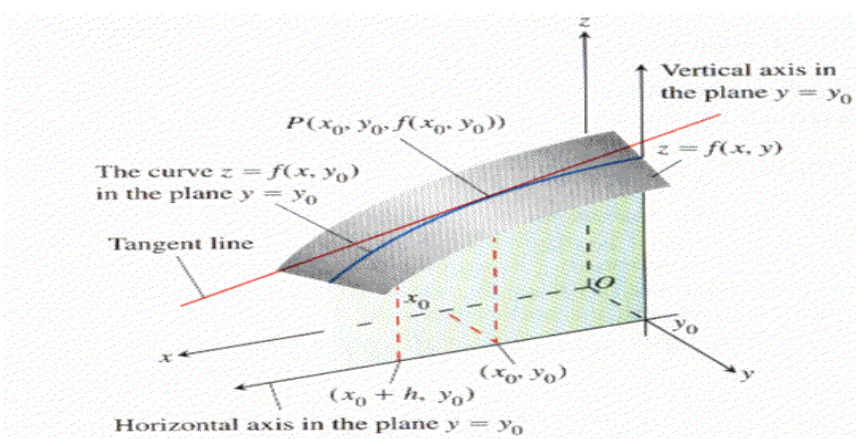
$$\Delta \mathbf{z} = \mathbf{f}\,(\mathbf{x}\,, \mathbf{y} + \Delta \mathbf{y}) - \mathbf{f}\,(\mathbf{x}, \mathbf{y})$$

If the ratio

$$\frac{\Delta z}{\Delta y} = \frac{f(x\,, y + \Delta y) - f(x, y)}{\Delta y}$$

approaches to a finite limit as $\Delta y \to 0$, then this limit is called Partial derivative of f with respect of y.

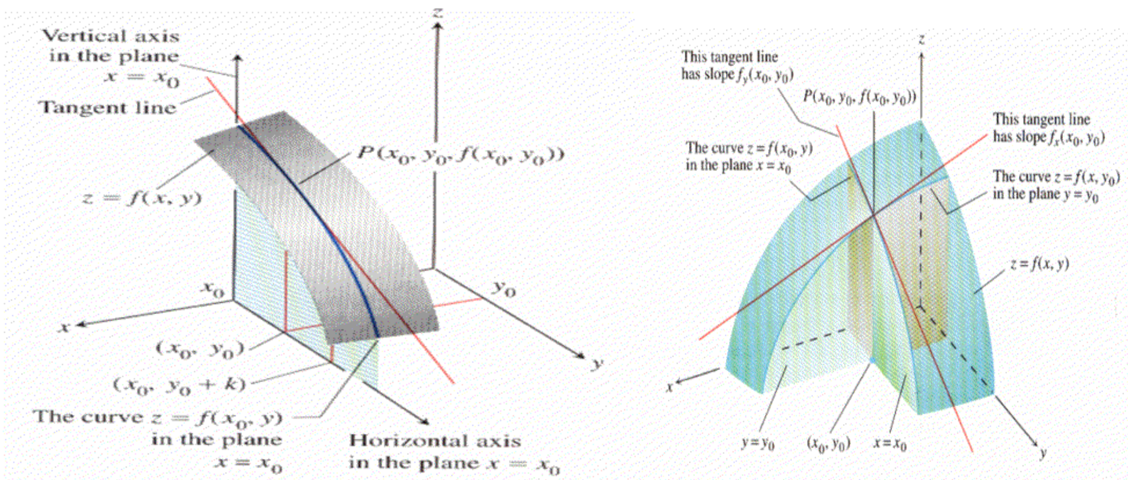
<u>Geometric Meaning of Partial Derivatives</u>



Suppose $z = f(x, y)$ is a function of two variables x and y. The graph of f is a surface. Let P be a point on the graph with coordinates $(x_0, y_0, f(x_0, y_0))$. If a point starting from P, changes its position on the surface such that y remains constant, then locus of this point is a curve of intersection of $z = f(x, y)$ and $y =$ constant. On this curve, $\frac{\partial z}{\partial x}$ is a derivative of $z = f(x, y)$ with respect to x with y constant.

Thus $\frac{\partial z}{\partial x} =$ the slope of the tangent to the curve at P

Similarly, $\frac{\partial z}{\partial y}$ is the gradient of the tangent at P to the curve of intersection of $z = f(x, y)$ and $x =$ constant. As shown in the figures given below:



Partial Derivatives of Higher Orders

The partial derivative f_x and f_y of function of f of two variables x and y, being functions of x and y, may possess derivatives. In such cases, the second order partial derivatives are defined as below:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(f_x) = (f_x)_x = f_{xx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(f_x) = (f_x)_y = f_{xy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(f_y) = (f_y)_x = f_{yx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(f_y) = (f_y)_y = f_{yy}$$

Thus, there are four second order partial derivatives for a function $z = f(x, y)$. The partial derivatives f_{xy} and f_{yx} are called mixed **second partials** and are **not equal** in general. Partial derivatives of order more than two can be defined in similar manner.

Example

$$z = \text{arc sin}\left(\frac{x}{y}\right).$$

$$\frac{\partial z}{\partial x} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\cdot\frac{1}{y} = \frac{1}{\sqrt{y^2-x^2}}$$

$$\frac{\partial z}{\partial y} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\cdot\frac{-x}{y^2} = \frac{-x}{y\sqrt{y^2-x^2}}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{-1}{2}(y^2-x^2)^{-3/2}2y \qquad = \frac{-y}{(y^2-x^2)^{3/2}}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$$= \frac{-1}{y\sqrt{y^2-x^2}} - \frac{x}{y}\left[\frac{x}{(y^2-x^2)^{3/2}}\right]$$

$$= \frac{-y^2+x^2-x^2}{y(y^2-x^2)^{3/2}} = \frac{-y}{(y^2-x^2)^{3/2}}$$

Hence

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

Example

$$f(x, y) = x\cos y + ye^x$$

$$\frac{\partial f}{\partial x} = \cos y + ye^x$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = ye^x$$

$$f(x, y) = x \cos y + y e^{x}$$
$$\frac{\partial f}{\partial y} = -x \sin y + e^{x}$$
$$\frac{\partial^2 f}{\partial x \partial y} = -\sin y + e^{x}$$
$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -x \cos y$$

Laplace's Equation

For a function $w = f(x, y, z)$, the equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

is called Laplace's equation.

Example

$$f(x, y) = e^{x} \sin y + e^{y} \cos x,$$
$$\frac{\partial f}{\partial x} = e^{x} \sin y - e^{y} \sin x$$
$$\frac{\partial^2 f}{\partial x^2} = e^{x} \sin y - e^{y} \cos x$$
$$\frac{\partial f}{\partial y} = e^{x} \cos y + e^{y} \cos x$$
$$\frac{\partial^2 f}{\partial y^2} = -e^{x} \sin y + e^{y} \cos x$$

Adding both partial second order derivative, we have

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = e^{x} \sin y - e^{y} \cos x$$
$$- e^{x} \sin y + e^{y} \cos x \quad = 0$$

Euler's theorem

The mixed derivative theorem

If f(x,y) and its partial derivatives f_x, f_y, f_{xy} and f_{yx} are defined throughout an open region containing a point (a, b) and are all continuous at (a, b), then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

Advantage of Euler's theorem

$$w = xy + \frac{e^y}{y^2 + 1}$$

The symbol $\frac{\partial^2 w}{\partial x \partial y}$ tell us to differentiate first with respect to y and then with respect to x. However, if we postpone the differentiation with respect to y and differentiate first with respect to x, we get the answer more quickly.

$$\frac{\partial w}{\partial x} = y \qquad \frac{\partial^2 w}{\partial y \partial x} = 1$$

Overview of lecture# 7

Chapter # 16 Partial derivatives

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