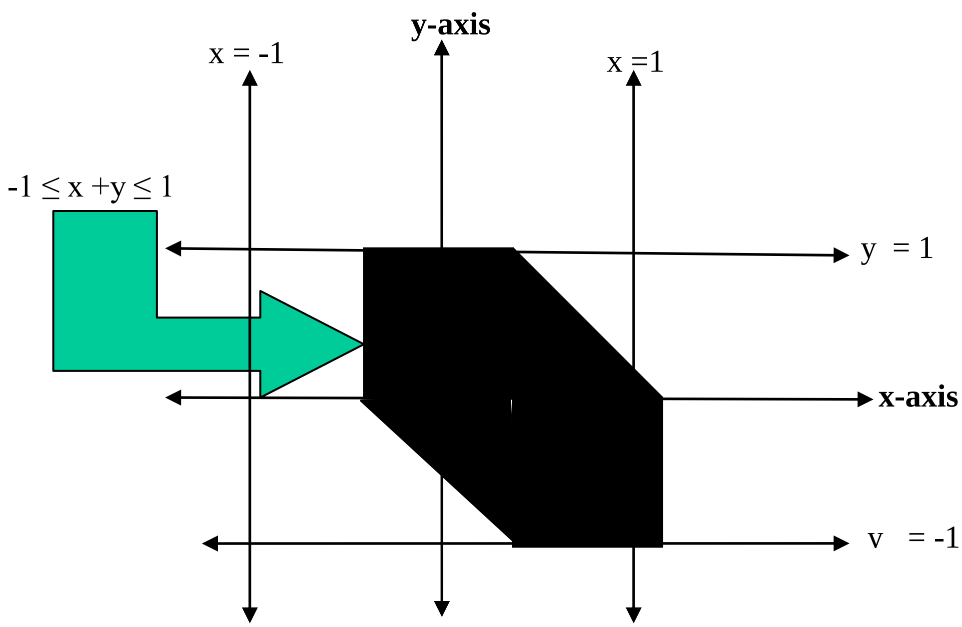
Lecture No-5 Limit of Multivariable Function

$f(x,y) = \sin^{-1}(x+y)$

Domain of f is the region in which $-1 \le x + y \le 1$



Domains and Ranges

Functions	Domain	Range
$\omega = \sqrt{x^2 + y^2 + z^2}$	Entire space	$[0, \infty)$
$\omega = \dfrac{1}{x^2 + y^2 + z^2}$	$(x,y,z) \neq (0,0,0)$	$(0, \infty)$
$\omega = xy \ln z$	Half space $z > 0$	$(-\infty, \infty)$

Examples of domain of a function

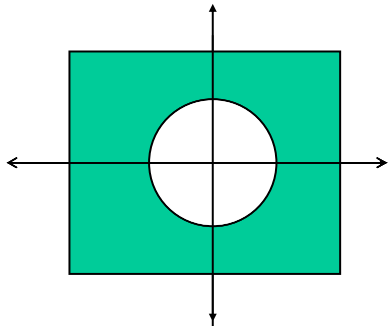
$f(x, y) = xy\sqrt{y - 1}$ Domain of f consists of region in xy plane where $y \geq 1$

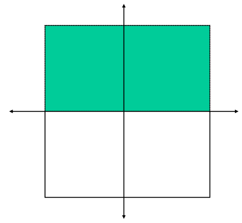
$f(x, y) = \sqrt{x^2 + y^2 - 4}$

Domain of f consists of region

in xy plane where $x^2 + y^2 \geq 4$

As shown in the figure



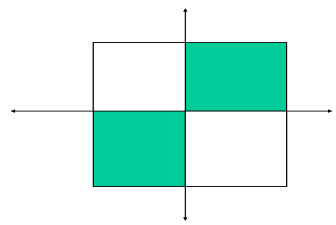


f(x, y) = lnxy

Domain of *f* consists of region lying in first and third quadrants in xy plane as shown in above figure right side.

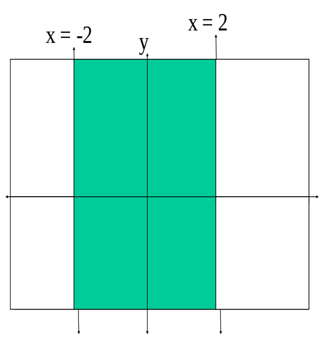
$f(x, y, z) = e^{xyz}$

Domain of f consists of region of three dimensional space



$$f(x, y) = \frac{\sqrt{4 - x^2}}{y^2 + 3}$$

Domain of f consists of region in xy plane $x^2 \leq 4$,$-2 \leq x \leq 2$



$$f(x, y, z) = \sqrt{25 - x^2 - y^2 - z^2}$$

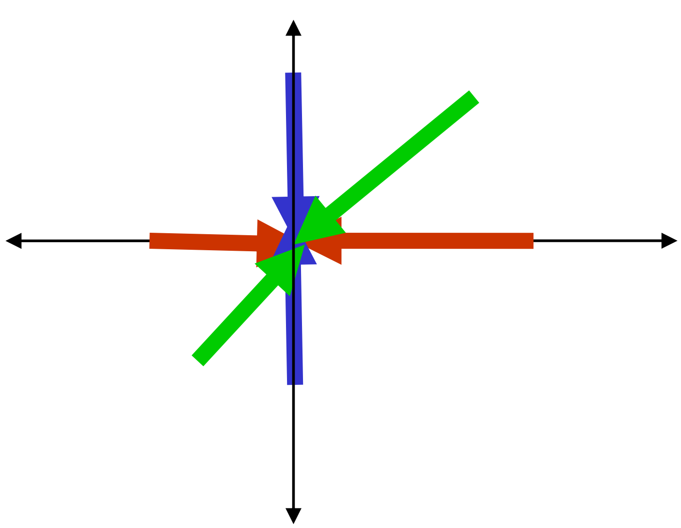
Domain of f consists of region in three dimensional space occupied by sphere centre at (0, 0, 0) and radius 5.

$$f(x, y) = \frac{x^3 + 2x^2y - xy - 2y^2}{x + 2y}$$

f(0,0) is not defined but we see that limit exits.

Approaching to (0,0) through x-axis	f (x,y)	Approaching to (0,0) through y-axis	f (x,y)
(0.5,0)	0.25	(0,0.1)	-0.1
(0.25,0)	0.0625	(0,0.001)	-0.001
(0.1,0)	0.01	(0,0.00001)	0.00001
(-0.25,0)	0.0625	(0,-0.001)	0.001
(-0.1,0)	0.01	(0,-0.00001)	0.00001

Approaching to (0,0) through y = x	f (x,y)
(0.5,0.5)	-0.25
(0.1,0.1)	-0.09
(0.01,0.01)	-0.0099
(-0.5,-0.5)	0.75
(-0.1,-0.1)	0.11
(-0.01,-0.01)	0.0101



Example

$$f(x,y) = \frac{xy}{x^2+y^2}$$

f(0,0) is not defined and we see that limit also does not exit.

Approaching to (0,0) through x-axis (y = 0)	f (x,y)	Approaching to (0,0) through y = x	f (x,y)
(0.5,0)	0	(0.5,0.5)	0.5
(0.1,0)	0	(0.25,0.25)	0.5
(0.01,0)	0	(0.1,0.1)	0.5
(0.001,0)	0	(0.05,0.05)	0.5
(0.0001,0)	0	(0.001,0.001)	0.5
(-0.5,0)	0	(-0.5,-0.5)	0.5
(-0.1,0)	0	(-0.25,-0.25)	0.5
(-0.01,0)	0	(-0.1,-0.1)	0.5
(-0.001,0)	0	(-0.05,-0.05)	0.5
(-0.0001,0)	0	(-0.001,-0.001)	0.5

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = 0 \text{ (along } y = 0)$$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = 0.5 \text{ (along } y = x)$$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} \text{ does not exist.}$$

Example

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

Let (x, y) approach $(0, 0)$ along the line $y = x$.

$$f(x,y) = \frac{xy}{x^2+y^2} = \frac{x \cdot x}{x^2+x^2} = \frac{1}{1+1} \qquad x \neq 0.$$

$$= \frac{1}{2}$$

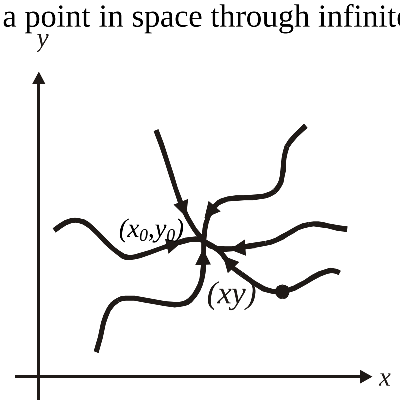
Let (x, y) approach $(0, 0)$ along the line $y = 0$.

$$f(x, y) = \frac{x.(0)}{x^2 + (0)^2} = 0 \quad , \quad x \neq 0.$$

Thus $f(x, y)$ assumes two different values as (x,y) approaches (0,0) along two different paths.

$$\lim_{(x, y) \to (0, 0)} f(x, y) \text{ does not exist.}$$

We can approach a point in space through infinite paths some of them are shown in the figure below.



Rule for Non-Existence of a Limit

If in

$$\lim_{(x, y) \to (a, b)} f(x, y)$$

We get *two or more different values as we approach (a, b) along different paths*, then

$$\lim_{(x, y) \to (a, b)} f(x, y)$$

does not exist.

The paths along which (a, b) is approached may be *straight lines or plane curves through (a, b).*

Example

$$\lim_{(x, y) \to (2, 1)} \frac{x^3 + x^2 y - x - y^2}{x + 2y}$$

$$= \frac{\lim_{(x, y) \to (2, 1)} (x^3 + 2x^2 y - xy - 2y^2)}{\lim_{(x, y) \to (2, 1)} (x + 2y)}$$

$$= \frac{\lim_{(x,y)\to(2,1)} (x^3 + 2x^2 y - x - y^2)}{\lim_{(x,y)\to(2,1)} (x + 2y)}$$

Example

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2 + y^2}}$$

We set

$$x = r\cos\theta,\; y = r\sin\theta$$

then $\dfrac{x}{\sqrt{x^2+y^2}} = \dfrac{r\cos\theta \,.\, r\sin\theta}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}}$

$= r\cos\theta\sin\theta$, **for** $r > 0$

Since $r = \sqrt{x^2+y^2}$, $r \to 0$ **as** $(x, y) \to (0, 0)$,

$$\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}} = \lim_{r\to 0} r\cos\theta\sin\theta = 0,$$

since $|\cos\theta\sin\theta| \leq 1$ **for all value of** θ.

RULES FOR LIMIT

If $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L_1$ *and* $\lim\limits_{(x,y)\to(x_0,y_0)} g(x,y) = L_2$ Then

(a) $\lim\limits_{(x,y)\to(x_0,y_0)} cf(x,y) = cL_1$ (if c is constant)

(b) $\lim\limits_{(x,y)\to(x_0,y_0)} \{f(x,y) + g(x,y)\} = L_1 + L_2$

$\lim\limits_{(x,y)\to(x_0,y_0)} \{f(x,y) - g(x,y)\} = L_1 - L_2$

(d) $\lim\limits_{(x,y)\to(x_0,y_0)} \{f(x,y)g(x,y)\} = L_1 L_2$

(e) $\lim\limits_{(x,y)\to(x_0,y_0)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L_1}{L_2}$ (if $L_2 = 0$)

$\lim\limits_{(x,y)\to(x_0,y_0)} c = c$ (c a constant), $\lim\limits_{(x,y)\to(x_0,y_0)} x_0 = x_0$, $\lim\limits_{(x,y)\to(x_0,y_0)} y_0 = y_0$

Similarly for the function of three variables.

Overview of lecture# 5

In this lecture we recall you all the limit concept which are prerequisite for this course and you can find all these concepts in the chapter # 16 (topic # 16.2)of your Calculus By Howard Anton.