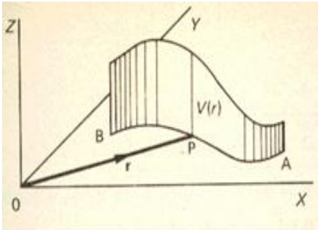
Lecture No -36 Scalar Field

Scalar Field

If a scalar field V(r) exists for all points on the curve ,

the $\sum_{p=1}^{n} V(r)dr_p$ with dr $\to 0$, defines the line integral

of V i.e line integral = $\int_c V(r)dr.$



We can illustrate this integral by erecting a continuous Ordinate to V(r) at each point of the curve $\int_c V(r)dr$ is then represented by the area of the curved surface between the ends A and B the curve c. To evaluate a line integral , the integrand is expressed in terms of x , y ,z with dr =dx + dy j + dz k

In practice , x , y and are often expressed in terms of parametric equation of a fourth variable (say u), i.e. x = x(u) ; y = y(u) ; z = z(u) . From these , dx, dy and dz can be written in terms of u and the integral evaluate in terms of this parameter u.

Example

If $V=xy^2z$, evaluate $\int_c V(r)dr$ along the curve c having parametric equations

$x = 3u; y=2u^2 ;z=u^3$ between A(0,0,0) and B(3,2,1)

$V= xy^2z = (3u)(4u^4)(u^3)=12u^8$

d**r**= dx**i**+ dy **j**+ dz **k** $\Rightarrow$ **dr= 3**du i +4udu j +$\mathbf{3u^2}$du k

for x = 3u ; $\therefore$ dx = 3du ; y = $2u^2$ $\therefore$ dy = 4u du ; z = u^3 $\therefore$ dz =$\mathbf{3u^2dz}$

Limiting : A(0,0,0) corresponds to B(3,2,1) corresponds to u

A(0,0,0) $\equiv$ u=0 ; B(3,2,1) $\equiv$ u = 1

$$\int_c V(r)dr = \int_0^1 12u^8(3\ i + 4u\ j + 3u^2\ k)du = \left| 36\frac{u^9}{9}i + 48\frac{u^{10}}{10}j + 36\frac{u^{11}}{11} \right|_0^1 = 4i + \frac{24}{5}j + \frac{36}{11}k$$

Example

If $V = xy + y^2z$ Evaluate $\int_c V(r)dr$ along the curve c defined by $x= t^2$; y = 2t ; z= t+5 between A(0,0,5) and B(4,4,7) . As before , expressing V and d**r** in term of the parameter t .

since $V=xy+y^2z$

$= (t^2)(2t)+(4t^2)(t+5)$

$= 6t^3 + 20t^2.$

$$\left.\begin{array}{ll} x = t^2 & dx = 2t\,dt \\ y = 2t & dy = 2\,dt \\ z = t+5 & dz = dt \end{array}\right\}$$

$\therefore$ dr = dx**i** + dy**j** + dz**k**

= 2t dt **i** + 2 dt **j** + dt**k**

$\therefore \int_C Vdr = \int_C (6t^3+20t^2)(\ 2t\ \mathbf{i} + 2\mathbf{j}+\mathbf{k})\ dt$

Limits: A (0, 0, 5) $\equiv$ t = 0;
B (4, 4, 7) $\equiv$ t = 2

$\therefore \int_C V dr = \int_0^2 (6t^3+20t^2)(2t\,\mathbf{i} + 2\,\mathbf{j} + \mathbf{k})\, dt$

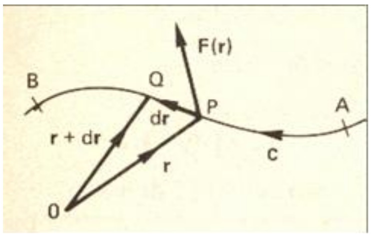
$\int_C V dr = 2\int_0^2 \{(6t^4+20t^3)\mathbf{i}+(6t^3+20t^2)\mathbf{j}$
$+(3t^3+10t^2)k\}dt.$

$= \frac{8}{15}(444\mathbf{i} + 290\mathbf{j} + 145\mathbf{k})$

Vector Field

If a vector field F(r) exist for all points of the curve c, then for each element of arc we can form the scalar product F.dr. Summing these products for all elements of arc, we have

$$\sum_{p=1}^{n} F.dr_p$$



The line integral of F(r) from A to B along the stated curve = $\int_C$ F.dr.

In this case, since **F**.d**r** is a scalar product, then the line integral is a scalar. To evaluate the line integral, F and d**r** are expressed in terms of x,y,z, and the curve in parametric form. We have

$\mathbf{F} = F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k}$

And $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$

Then $\mathbf{F}.d\mathbf{r} = (F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k}).(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}) = \int_c (F_1 dx + F_2 dy + F_3 dz)$

Now for an example to show it in operation.

Example

If $F(r) = x^2y\,\mathbf{i} + xz\,\mathbf{j} + 2yz\,\mathbf{k}$ **,** Evaluate $\int_c F.dr$ between A(0,0,0) and B(4,2,1) along the curve c having parametric equations x=4t ; $y-2t^2$; $z = t^3$

Expressing everthing in terms of the parameter t, we have

$dx = 4\,dt$; $dy = 4t dt$; $dz = 3t^2\,dt$

$x^2y = (16t^2)(2t^2) = 32\,t^4$

$x = 4t \quad \therefore \quad dx = 4\,dt$

$xz = (4t)(t^3) = 4\,t^4$

$y = 2\,t^2 \quad dy = 4t\,dt$

$2yz = (4\,t^2)(t^3) = 4\,t^5$

$z = t^3 \quad \therefore\ dz = 3t^2\,dt$

$\mathbf{F} = 32\,t^4\,\mathbf{i} + 4\,t^4\,\mathbf{j} - 4\,t^5\,\mathbf{k}$

$dr = 4dt\,\mathbf{i} + 4t\,dt\,\mathbf{j} + 3t^2\,\mathbf{k}$

Then $\int \mathbf{F}.d\mathbf{r} = \int (32t^4\mathbf{i}+4t^4\mathbf{j}-4t^5\mathbf{k}).$
$(4dt\,\mathbf{i} + 4t\,dt\,\mathbf{j} + 3t^2\,dt\,\mathbf{k})$
$= \int (128t^4+16t^5+12t^7)\,dt$

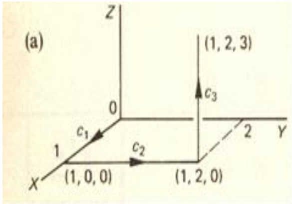
Limits: A(0,0,0) $\equiv$ t = 0;
B (4, 2, 1) $\equiv$ t = 1

$\int_C \mathbf{F}.d\mathbf{r} = (128t^4 + 16t^5 + 12t^7)dt = \frac{128}{5}t^5 + \frac{16}{6}t^6 + \frac{12}{8}t^8 = \frac{128}{5} + \frac{8}{3} + \frac{3}{2} = 29.76$

Example

If $\mathbf{F}(\mathbf{r}) = x^2y\mathbf{i} + 2yz\mathbf{j} + 3z^2x\mathbf{k}$

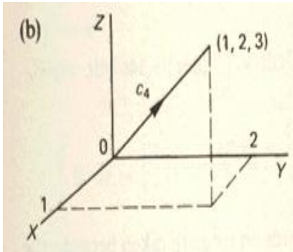
Evaluate $\int_C \mathbf{F}.d\mathbf{r}$ between A(0,0,0) and B(1,2,3)



B (1, 2, 3)

(a) along the straight line
c_1 from (0, 0, 0) to (1, 0, 0)
then c_2 from (1, 0, 0) to (1, 2, 0)
and c_3 from (1, 2, 0) to (1, 2, 3)

(b) along the straight line c_4 joining (0, 0, 0) to (1, 2, 3).



We first obtain an expression for $\mathbf{F}.d\mathbf{r}$ which is

$$\mathbf{F}.d\mathbf{r} = (x^2y\mathbf{i} + 2yz\mathbf{j} + 3z^2x\mathbf{k}).(dx\,\mathbf{i} + dy\mathbf{j} + dz\,\mathbf{k})$$

$$\mathbf{F}.d\mathbf{r} = x^2y\,dx + 2yz\,dy + 3z^2x\,dz$$

$$\int \mathbf{F}.d\mathbf{r} = \int x^2ydx + \int 2yzdy + \int 3z^2xdz$$

Here the integration is made in three sections, along c_1, c_2 and c_3.

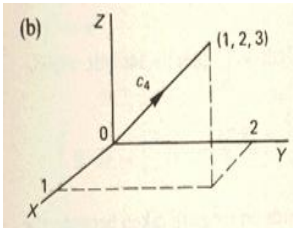
(i) c_1: $y = 0,\ z = 0,\ dy = 0,\ dz = 0$

$$\therefore \int_{C_1} \mathbf{F}.d\mathbf{r} = 0+0+0 = 0$$

(ii) c_2: The conditions along c_2 are

c_2: $x = 1,\ z = 0,\ dx = 0,\ dz = 0$

$$\therefore \int_{C_2} \mathbf{F}.d\mathbf{r} = 0 + 0 + 0 = 0$$



(iii) c_3: $x = 1,\ y = 2,\ dx = 0,\ dy = 0$

$$\int_{C_3} \mathbf{F}.d\mathbf{r} = 0 + 0 + \int_0^3 3z^2dz = 27$$

Summing the three partial results

$$\int_{(0,0,0)}^{(1,2,3)} \mathbf{F}.d\mathbf{r} = 0 + 0 + 27 = 27$$

$$\therefore \int_{c_1+c_2+c_3} \mathbf{F}.d\mathbf{r} = 27$$

If t taken as the parameter, the parametric equation of c are $x = t;\ y = 2t;\ z = 3t$

$(0, 0, 0) \Rightarrow t = 0$, $(1, 2, 3) \Rightarrow t = 1$ and the limits of t are $t = 0$ and $t = 1$

$$\mathbf{F} = 2t^3\mathbf{i} + 12t^2\mathbf{j} + 27t^3\mathbf{k}$$

$$d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + \mathbf{k}dz = dt\,\mathbf{i} + 2dt\,\mathbf{j} + 3dt\,\mathbf{k}$$

$$\int_{C_4} \mathbf{F}.d\mathbf{r} = \int_0 (2t^3\mathbf{i}+12t^3\mathbf{j}+27t^3\mathbf{k}).(\mathbf{i}+2\mathbf{j}+3\mathbf{k})\,dt = \int_0^1 (2t^3+24t^2+81t^3)dt$$

$$= \int_0^1 (83t^3 + 24t^2)\,dt = \left[83\frac{t^4}{4} + 8t^3\right]_0^1 = \frac{115}{4} = 28.75$$

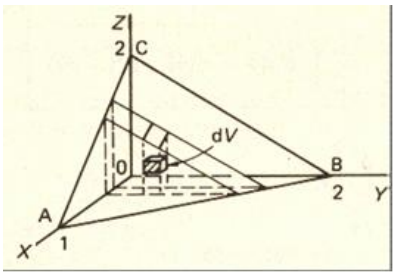
So the value of the line integral depends on the path taken between the two end points A and B

(a) $\int \mathbf{F.dr}$ via c_1, c_2 and c_3 = 27

(b) $\int \mathbf{F.dr}$ via c_4 = 28.75

Example

Evaluate $\int_v F\, dv$ where V is the region bounded by the planes x = 0 ,y = 0 , z = 0 and 2x + y = 2 , and F = 2z **i** +y **k.** To sketch the surface 2x + y + z = 2, note that



when z = 0, 2x+y=2 i.e. y = 2 − 2x
when y = 0, 2x+z=2 i.e. z = 2 − 2x
when x = 0, y+z=2 i.e. z = 2 − y
Inserting these in the planes
x = 0, y = 0, z = 0 will help.
The diagram is therefore.
So 2x + y + z = 2 cuts the axes at
A(1,0,0); B (0, 2, 0); C (0, 0, 2).
Also F = 2z**i** + y**k**;
z = 2 – 2x – y = 2 (1 – x) – y

$$\therefore \int_v F dV = \int_0^1 \int_0^{2(1-x)} \int_0^{2(1-x)-y} (2x\mathbf{i}+y\mathbf{k})\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^{2(1-x)} \left[z^2\mathbf{i}+yz\mathbf{k}\right]_{z=0}^{z=2(1-x)-y} dy\,dx$$

$$= \int_0^1 \int_0^{2(1-x)} \{[4(1-x)^2 - 4(1-x)y + y^2]\mathbf{i} + [2(1-x)y - y^2]k\}\,dy\,dx$$

$$\int_v F dV = \frac{1}{3}\ (2\mathbf{i} + \mathbf{k})$$