Lecture No -35 Definite Integrals

Definite integral for $\sin^n x$ and $\cos^n x$, $0 \le x \le \pi/2$

$$\int_0^{\frac{\pi}{2}} \sin^2 x dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1-\cos 2x) dx = \frac{1}{2}\left| x - \frac{\sin 2x}{2} \right|_0^{\frac{\pi}{2}} = \frac{1}{2}\left| \frac{\pi}{2} - \frac{\sin \pi}{2} \right| = \frac{\pi}{4}$$

$$\int_0^{\frac{\pi}{2}} \sin^2 x dx = \frac{1}{2}\frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} \cos^2 x dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\cos 2x) dx = \frac{1}{2}\left| x + \frac{\sin 2x}{2} \right|_0^{\frac{\pi}{2}} = \frac{1}{2}\left| \frac{\pi}{2} + \frac{\sin \pi}{2} \right| = \frac{\pi}{4}$$

$$\int_0^{\frac{\pi}{2}} \cos^2 x dx = \frac{1}{2}\frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} \sin^3 x dx = \int_0^{\frac{\pi}{2}} \sin^2 x \sin x dx = \int_0^{\frac{\pi}{2}} (1-\cos^2 x)\sin x dx = \int_0^{\frac{\pi}{2}} \sin x dx + \int_0^{\frac{\pi}{2}} \cos^2 x(-\sin x) dx$$

$$= \left|-\cos x\right|_0^{\frac{\pi}{2}} + \left|\frac{\cos^3 x}{3}\right|_0^{\frac{\pi}{2}} = -\cos\frac{\pi}{2} + \cos 0 + \frac{1}{3}\left[\cos^3\frac{\pi}{2} - \cos^3 0\right] = 1 - \frac{1}{3} = \frac{2}{3}$$

$$\int_0^{\frac{\pi}{2}} \cos^3 x dx = \int_0^{\frac{\pi}{2}} \cos^2 x \cos x dx = \int_0^{\frac{\pi}{2}} (1-\sin^2 x)\cos x dx = \int_0^{\frac{\pi}{2}} \cos x dx - \int_0^{\frac{\pi}{2}} \sin^2 x(\cos x) dx$$

$$= \left|\sin x\right|_0^{\frac{\pi}{2}} - \left|\frac{\sin^3 x}{3}\right|_0^{\frac{\pi}{2}} = \sin\frac{\pi}{2} - \sin 0 - \frac{1}{3}\left[\sin^3\frac{\pi}{2} - \sin^3 0\right] = 1 - \frac{1}{3} = \frac{2}{3}$$

$$\int_0^{\frac{\pi}{2}} \sin^4 x dx = \int_0^{\frac{\pi}{2}} (\sin^2 x)^2 dx = \int_0^{\frac{\pi}{2}} \left[\frac{1-\cos 2x}{2}\right]^2 dx = \frac{1}{4}\int_0^{\frac{\pi}{2}} (1 - 2\cos 2x + \cos^2 2x) dx$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} (1 - 2\cos 2x + \frac{1+\cos 4x}{2}) dx = \frac{1}{4}\int_0^{\frac{\pi}{2}} (\frac{3}{2} - 2\cos 2x + \frac{c\,os\, 4x}{2}) dx$$

$$= \frac{1}{4}\left|\frac{3}{2}x - \sin 2x + \frac{\sin 4x}{8}\right|_0^{\frac{\pi}{2}} = \frac{1}{4}\left[\frac{3}{2}\frac{\pi}{2} - \sin\pi + \frac{\sin 2\pi}{8}\right]$$

$$\int_0^{\frac{\pi}{2}} \sin^4 x dx = \frac{1}{4}\left[\frac{3}{2}\frac{\pi}{2}\right] \qquad so \qquad \int_0^{\frac{\pi}{2}} \sin^4 x dx = \frac{3}{4}\frac{1}{2}\frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} \cos^4 x dx = \int_0^{\frac{\pi}{2}} (\cos^2 x)^2 dx = \int_0^{\frac{\pi}{2}} \left[\frac{1+\cos 2x}{2}\right]^2 dx = \frac{1}{4}\int_0^{\frac{\pi}{2}} (1+2\cos 2x + \cos^2 2x)dx$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} (1+2\cos 2x + \frac{1+\cos 4x}{2})dx = \frac{1}{4}\int_0^{\frac{\pi}{2}} (\frac{3}{2}+2\cos 2x + \frac{c\,os 4x}{2})dx$$

$$= \frac{1}{4}\left|\frac{3}{2}x + \sin 2x + \frac{\sin 4x}{8}\right|_0^{\frac{\pi}{2}} = \frac{1}{4}\left[\frac{3}{2}\frac{\pi}{2} + \sin\pi + \frac{\sin 2\pi}{8}\right]$$

$$\int_0^{\frac{\pi}{2}} \cos^4 x dx = \frac{1}{4}\left[\frac{3}{2}\frac{\pi}{2}\right] \qquad \text{So} \qquad \int_0^{\frac{\pi}{2}} \cos^4 x dx = \frac{3}{4}\frac{1}{2}\frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} \sin^5 x dx = \frac{4}{5}\frac{2}{3} \qquad \text{and} \qquad \int_0^{\frac{\pi}{2}} \cos^5 x dx = \frac{4}{5}\frac{2}{3}$$

$$\int_0^{\frac{\pi}{2}} \sin^6 x dx = \frac{5}{6}\frac{3}{4}\frac{1}{2}\frac{\pi}{2} \qquad \text{and} \qquad \int_0^{\frac{\pi}{2}} \cos^6 x dx = \frac{5}{6}\frac{3}{4}\frac{1}{2}\frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} \sin^7 x dx = \frac{6}{7}\frac{4}{5}\frac{2}{3} \qquad \text{and} \qquad \int_0^{\frac{\pi}{2}} \cos^7 x dx = \frac{6}{7}\frac{4}{5}\frac{2}{3}$$

$$\int_0^{\frac{\pi}{2}} \sin^8 x dx = \frac{7}{8}\frac{5}{6}\frac{3}{4}\frac{1}{2}\frac{\pi}{2} \qquad \text{and} \qquad \int_0^{\frac{\pi}{2}} \cos^8 x dx = \frac{7}{8}\frac{5}{6}\frac{3}{4}\frac{1}{2}\frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} \sin^9 x dx = \frac{8}{9}\frac{6}{7}\frac{4}{5}\frac{2}{3} \qquad \text{and} \qquad \int_0^{\frac{\pi}{2}} \cos^9 x dx = \frac{8}{9}\frac{6}{7}\frac{4}{5}\frac{2}{3}$$

$$\int_0^{\frac{\pi}{2}} \sin^{10} x dx = \frac{9}{10}\frac{7}{8}\frac{5}{6}\frac{3}{4}\frac{1}{2}\frac{\pi}{2} \qquad \text{and} \qquad \int_0^{\frac{\pi}{2}} \cos^8 x dx = \frac{9}{10}\frac{7}{8}\frac{5}{6}\frac{3}{4}\frac{1}{2}\frac{\pi}{2}$$

<u>Wallis Sine Formula</u>

When n is even $\int_0^{\frac{\pi}{2}} \sin^n x dx = \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}.\frac{n-7}{n-6}.--------\frac{5}{6}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}$

When n is odd $\int_0^{\frac{\pi}{2}} \cos^n x dx = \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}.\frac{n-7}{n-6}.--------\frac{6}{7}.\frac{4}{5}.\frac{2}{3}.$

$$\int_0^{\frac{\pi}{2}} \sin^{11} x dx = \frac{10.8.6.4.2}{11.9.7.5.3} \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \cos^{11} x dx = \frac{10.8.6.4.2}{11.9.7.5.3}$$

$$\int_0^{\frac{\pi}{2}} \sin^{12} x dx = \frac{11.9.7.5.3.1}{10.8.6.4.2}\frac{\pi}{2} \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \cos^{12} x dx = \frac{11.9.7.5.3.1}{10.8.6.4.2}\frac{\pi}{2}$$

Integration By Parts

$$\int UVdx = U\int V\,dx - \int\left[\int Vdx.\frac{dU}{dx}\right]dx$$

Example *Evaluate* $\int$ x lnx dx

$\int$ x lnx dx= lnx $\int$ xdx – $\int$ [$\int$ x dx.$\frac{d}{dx}$(ln x)] dx (We are integrating by parts)

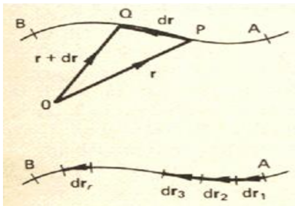
= ln x($\frac{x^2}{2}$)-$\int$ ($\frac{x^2}{2}$)($\frac{1}{x}$)dx=($\frac{x^2}{2}$) ln x-$\int$ ($\frac{x}{2}$)d x = ($\frac{x^2}{2}$) ln x-$\frac{1}{2}$ ($\frac{x^2}{2}$)

Example *Evaluate* $\int$ x sinx dx

$\int$ **x** sinx dx =x$\int$ sin xdx – $\int$ [$\int$ sinx dx.$\frac{d}{dx}$(x)] dx (We are integrating by parts)

= x(-cosx)-$\int$ (-cosx)(1)dx= -x(cosx) +$\int$ cosx d x = -x(cosx)+sin x

Line Integrals

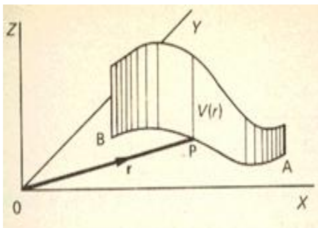


Let a point p on the curve c joining A and B be denoted by the position vector **r** with respect to origin O. If q is a neighboring point on the curve with position vector **r+dr** , then $\overline{PQ}$ = **r**

The curve c can be divided up into many n such small arcs , approximating to d**r**$_1$, d**r**$_2$, d**r**$_3$,………. d**r**$_p$,……

so that $\overline{AB}\sum_{p=1}^{n} dr_p$ where d**r**$_p$ is a vector representing the element of the arc in both magnitude and direction. If dr $\rightarrow$ 0 , then the length of the curve AB=$\int_c dr$.

Scalar Field



If a scalar field V(r) exists for all points on the curve ,

the $\sum_{p=1}^{n} V(r)dr_p$ with dr $\rightarrow$ 0 , defines the line integral

of V i.e line integral = $\int_c V(r)dr.$

We can illustrate this integral by erecting a continuous Ordinate to V(r) at each point of the curve $\int_c V(r)dr$ is then represented by the area of the curved surface between the ends A and B the curve c. To evaluate a line integral , the integrand is expressed in terms of x , y ,z with dr =dx + dy j + dz k

In practice , x , y and are often expressed in terms of parametric equation of a fourth variable (say u), i.e. x = x(u) ; y = y(u) ; z = z(u) . From these , dx, dy and dz can be written in terms of u and the integral evaluate in terms of this parameter u.