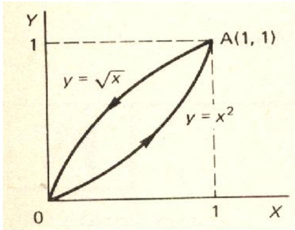
Lecture -34………Examples

Example

Evaluate the line integral $I = \oint_C \{xy\,dx + (2x - y)\,dy\}$ round the region bounded by the curves $y = x^2$ and $x = y^2$ by the use of Green's theorem.



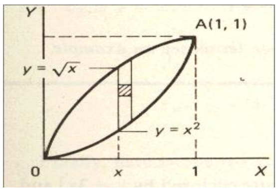
Points of intersection are O(0, 0) and A(1, 1).

$$I = \oint_C \{xy\,dx + (2x - y)\,dy\}$$

$$\oint_C \{P dx + Q dy\} = -\iint_R \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dx\,dy$$

$$P = xy \quad \therefore \frac{\partial P}{\partial y} = x; \quad Q = 2x - y \quad \therefore \frac{\partial Q}{\partial x} = 2$$

$$I = -\iint_R (x-2)\,dx\,dy = -\int_0^1 \int_{y=x^2}^{y=\sqrt{x}} (x-2)\,dy\,dx$$



$$= -\int_0^1 (x-2)\,[\,y\,]_{x^2}^{\sqrt{x}}\,dx$$

$$\therefore\ I = -\int_0^1 (x-2)\left(\sqrt{x} - x^2\right) dx = -\int_0^1 (x^{3/2} - x^3 - 2x^{1/2} + 2x^2)\,dx$$

$$= -\left[\frac{2}{5}x^{5/2} - \frac{1}{4}x^4 - \frac{4}{3}x^{3/2} + \frac{2}{3}x^3\right]_0^1 = \frac{31}{60}$$

In this special case when P=y and Q= – x so $\frac{\partial P}{\partial y} = 1$ and $\frac{\partial Q}{\partial x} = -1$

Green's theorem then states $\iint_R \{1 - (-1)\}\,dx\,dy = -\oint_C (P\,dx + Q\,dy)$

i.e. $2\iint_R dx\,dy = -\oint_C (y\,dx - x\,dy) = \oint_C (x\,dy - y\,dx)$

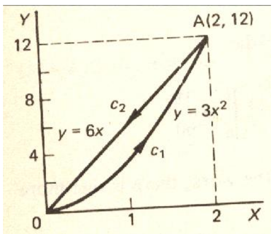
Therefore, the area of the closed region $A = \iint_R dx\,dy = \frac{1}{2}\oint_C (x\,dy - y\,dx)$

Example

Determine the area of the figure enclosed by $y = 3x^2$ and $y = 6x$.

Points of intersection : $3x^2 = 6x \quad \therefore\ x = 0$ or 2

Area $A = \frac{1}{2}\oint_C (x\,dy - y\,dx)$



We evaluate the integral in two parts, i.e.
OA along c_1 and AO along c_2

$$2A = \int_{c_1 \text{ (along OA)}} (xdy - ydx) + \int_{c_2 \text{ (along OA)}} (xdy - ydx) = I_1 + I_2$$

I_1: c_1 is $y = 3x^2 \quad \therefore\ dy = 6x\,dx$

$$\therefore \quad I_1 = \int_0^2 (6x^2dx - 3x^2dx) = \int_0^2 3x^2\,dx = \left[\,x^3\,\right]_0^2 = 8 \;\therefore\; I_1 = 8$$

Similarly, for c_2 is $y = 6x \quad \therefore \; dy = 6\,dx$

$$\therefore \quad I_2 = \int_2^0 (6x\,dx - 6x\,dx) = 0$$

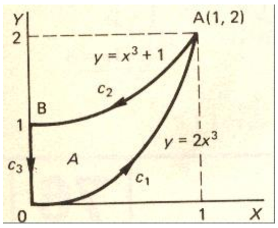
$\therefore \quad I_2 = 0$

$\therefore \quad I = I_1 + I_2 = 8 + 0 = 8$

$\therefore \; A = 4$ square units

Example

Determine the area bounded by the curves $y = 2x^3$, $y = x^3 + 1$ and the axis $x = 0$ for $x \geq 0$.



Here it is $y = 2x^3$; $y = x^3 + 1$; $x = 0$

Point of intersection $2x^3 = x^3 + 1 \;\therefore\; x^3 = 1 \;\therefore\; x = 1$

Area $A = \dfrac{1}{2}\oint_C (x\,dy - y\,dx) \;\therefore\; 2A = \oint_C (x\,dy - y\,dx)$

(a) OA : c_1 is $y = 2x^3 \;\therefore\; dy = 6x^2\,dx$

$$\therefore I_1 = \int_{c_1} (xdy - ydx) = \int_0^1 (6x^3dx - 2x^3\,dx) \quad = \int_0^1 4x3\,dx = \left[\,x^4\,\right]_0^1 = 1$$

$\therefore I_1 = 1$

(b) AB: c_2 is $y = x^3 + 1 \;\therefore\; dy = 3x^2\,dx$

$$\therefore \quad I_2 = \int_1^0 \{3x^3\,dx - (x^3 + 1)\,dx\} = \int_1^0 (2x^3 - 1)\,dx = \left[\frac{x^4}{2} - x\right]_1^0 = -\left(\frac{1}{2} - 1\right) = \frac{1}{2}$$

$\therefore \quad I_2 = \dfrac{1}{2}$

(c) BO: c_3 is $x = 0 \quad \therefore \; dx = 0$

$$I_3 = \int_{y=1}^{y=0} (xdy - ydx) = 0 \quad \therefore\; I_3 = 0$$

$$\therefore \quad 2A = I = I_1 + I_2 + I_3 = 1 + \frac{1}{2} + 0 = 1\frac{1}{2} \quad \therefore\; A = \frac{3}{4} \text{ square units}$$

Revision Summary

Properties of line integrals

- Sign of line integral is reversed when the direction of integration along the path is reversed.
- Path of integration parallel to y-axis, $dx = 0 \;\therefore\; I_c = \int_c Q\,dy$.
- Path of integration parallel to x-axis, $dy = 0 \;\therefore\; I_c = \int_c P\,dx$.
- Path of integration must be continuous and single-valued.
- Dependence of line integral on path of integration.
- In general, the value of the line integral depends on the particular path of integration.
- Exact differential
  If $P\,dx + Q\,dy$ is an exact differential

(a) $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

(b) $I = \int_c$ (P dx + Q dy) is independent of the path of integration

(c) $I = \oint_C$ (P dx + Q dy) is zero.

- Exact differential in three variables.
  If P dx + Q dy + R dw is an exact differential
  (a) $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$; $\frac{\partial P}{\partial w} = \frac{\partial R}{\partial x}$; $\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial w}$
  (b) $\int_c$ (P dx + Q dy + R dw) is independent of the path of integration.
  (c) $\oint_C$ (P dx + Q dy + R dw) is zero.
- Green's theorem

$\oint_C (P\,dx+Q\,dy) = -\iint_R \left\{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right\} dx\,dy$ and, for a simple closed curve,

$\oint_C (x\,dy - y\,dx) = 2\iint_R dx\,dy = 2A$

where A is the area of the enclosed figure.

Gradient of a scalar function

Del operator is given by $\nabla = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)$

$\nabla\phi = \text{grad}\ \phi = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\phi = \mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}$

$\text{grad}\phi = \nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$

Div (Divergence of a vector function)

If $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

then div A = $\nabla.A = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right).\left(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}\right)$

$\therefore$ div $\mathbf{A} = \nabla.\mathbf{A} = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z}$

Note that

(a) the grad operator ∇ acts on a scalar and gives a vector

(b) the div operator $\nabla.$ acts on a vector and gives a scalar.

Example

If $A = x^2y\mathbf{i} - xyz\mathbf{j} + yz^2\mathbf{k}$ then

Div $\mathbf{A} = \nabla.\mathbf{A} = \frac{\partial}{\partial x}(x^2y) - \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(yz^4) = 2xy - xz + 2yz$

Example

If A = $2x^2y\mathbf{i} - 2(xy^2+y^3)\mathbf{j}+3y^2z^2\mathbf{k}$ determine **∇.A** i.e. div **A**.

A= $2x^2y\mathbf{i} - 2(xy^2 + y^3z)\mathbf{j} + 3y^2z^2\mathbf{k}$

$$\nabla.\mathbf{A} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} = 4xy - 2(2xy+3y^2z)+6y^2z = 4xy - 4xy - 6y^2z+6y^2z = 0$$

Such a vector A for which **∇.A** = 0 at all points, i.e. for all values of x, y, z, is called a solenoid vector. It is rather a special case.

Curl (curl of a vector function)

The curl operator denoted by ∇×, acts on a vector and gives another vector as a result.

If **A**= $a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ then curl **A=∇×A**.

i.e. curl $\mathbf{A}=\nabla\times\mathbf{A}= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\times \left(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}\right)$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$\therefore\ \nabla\times\mathbf{A} = \mathbf{i}\left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right) + \mathbf{j}\left(\frac{\partial a_1}{\partial z} - \frac{\partial a_3}{\partial x}\right) + \mathbf{k}\left(\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y}\right)$$

Curl A is thus a vector function.

Example

If $\mathbf{A}=(y^4-x^2z^2)\mathbf{i}+(x^2+y^2)\mathbf{j}-x^2yzk$,determine curl A at the point (1,3, −2).

$$\text{Curl } \mathbf{A} = \nabla\times\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y4-x^2z^2 & x^2+y^2 & -x^2yz \end{vmatrix}$$

Now we expand the determinant

$$\nabla\times\mathbf{A} = \mathbf{i}\left\{\frac{\partial}{\partial y}(-x^2yz) - \frac{\partial}{\partial z}(x^2+y^2)\right\} - \mathbf{j}\left\{\frac{\partial}{\partial x}(-x^2yz) - \frac{\partial}{\partial z}(y^4 - x^2z^2)\right\}$$
$$+ \mathbf{k}\left\{\frac{\partial}{\partial x}(x^2 + y^2) - \frac{\partial}{\partial y}(y^4 - x^2z^2)\right\}$$

$\nabla\times\mathbf{A}=\mathbf{i}\{-x^2z\}-\mathbf{j}\{-2xyz+2x^2z\}+\mathbf{k}(2x-4y^3)$. ∴ At (1, 3, − 2),

$\nabla\times\mathbf{A} = \mathbf{i}\,(2)-\mathbf{j}\,(12 - 4) + \mathbf{k}\,(2 - 108) = 2\mathbf{i} - 8\mathbf{j} - 106\mathbf{k}$

Example

Determine curl F at the point (2,0,3) given that $\mathbf{F}=ze^{2xy}\mathbf{i}+2xycosy\mathbf{j}+(x+2y)\mathbf{k}$.

$$\text{In determine form, curl } \mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ze^{2xy} & 2xzcosy & x+2y \end{vmatrix}$$

Now expand the determinant and substitute the values for x, y and z, finally obtaining curl

$\nabla\times\mathbf{F} = \mathbf{i}\{2 - 2x\cos y\} - \mathbf{j}\{1 - e^{2xy}\} + \mathbf{k}\,(\{2z\cos y - 2xze^{2xy}\}$

∴At(2,0,3) $\nabla\times\mathbf{F}=\mathbf{i}(2-4)-\mathbf{j}(1-1)+\mathbf{k}(6-12) = -2\mathbf{i}-6\mathbf{k} = -2\,(\mathbf{i} + 3k)$

Summary of grad, div and curl

(a) Grad operator ∇ acts on a scalar field to give a vector field.
(b) Div operator ∇**.** Acts o a vector field to give a scalar field.
(c) Curl operator $\nabla \times$ acts on a vector field to give a vector field.
(d) With a scalar function ϕ (x,y,z)

$$\text{Grad } \phi = \nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i}+\frac{\partial\phi}{\partial y}\mathbf{j}+\frac{\partial\phi}{\partial z}\mathbf{k}$$

(e) With a vector function $A=a_x i+a_y\mathbf{j}+a_z\mathbf{k}$

(i) $\text{div } \mathbf{A} = \nabla.\mathbf{A}= \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$

(ii) $\text{Curl } \mathbf{A} = \nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}$

Multiple Operations

We can combine the operators grad, div and curl in multiple operations, as in the examples that follow.

EXAMPLE

If $A = x^2y\mathbf{i} + yz^3\mathbf{j} - zx^3\mathbf{k}$

Then $\text{div } \mathbf{A}=\nabla.\mathbf{A} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y}+ \mathbf{k}\frac{\partial}{\partial z}\right).(x^2y\mathbf{i} + yz^3\mathbf{j} - zx^3\mathbf{k})$

$= 2xy+z^3 + x^3 = \phi \text{ (say)}$

Then $\text{grad (div } \mathbf{A}) = \nabla(\nabla.\mathbf{A}) = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} = (2y+3x^2)\mathbf{i}+(2x)\mathbf{j}+(3z^2)\mathbf{k}$

i.e., $\text{grad div } A = \nabla(\nabla.\mathbf{A})= (2y+3x^2)\mathbf{i}+2x\mathbf{j}+3z^2\mathbf{k}$

Example

If $\phi = xyz - 2y^2z + x^2z^2$ determine div grad ϕ at the point (2, 4, 1).
First find grad ϕ and then the div of the result.
$\text{div grad } \phi = \nabla.(\nabla\phi)$
We have $\phi = xyz - 2y^2z + x^2z^2$

$\text{grad } \phi = \nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} =(yz+2xz^2)\mathbf{i}+(xz-4yz)\mathbf{j}+(xy-2y^2+2x^2z)\mathbf{k}$

$\therefore \text{div grad } \phi=\nabla.(\nabla\phi) = 2z^2 - 4z+2x^2$

$\therefore$ At (2,4,1), $\text{div grad } \phi = \nabla.(\nabla\phi) = 2 - 4 + 8 = 6$

$\text{grad } \phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$

Then $\text{div grad } \phi = \nabla.(\nabla\phi)=\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y}+ \mathbf{k}\frac{\partial}{\partial z}\right).\left(\frac{\partial\phi}{\partial x}\mathbf{i}+\frac{\partial\phi}{\partial y}\mathbf{j}+\frac{\partial\phi}{\partial z}\mathbf{k}\right)= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} + \frac{\partial^2\phi}{\partial z^2}$

$\therefore \text{div grad } \phi=\nabla.(\nabla\phi) = \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}$

Example

If F = $x^2yz\mathbf{i} + xyz^2\mathbf{j} + y^2z\mathbf{k}$ determine curl **F** at the point (2, 1, 1).Determine an expression for curl F in the usual way, which will be a vector, and then the curl of the result. Finally substitute values.

Curl curl $\mathbf{F}=\nabla\times(\nabla\times\mathbf{F}) = \mathbf{i}+2\mathbf{j}+6\mathbf{k}$

$$\text{For curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & xyz^2 & y^2z \end{vmatrix} = (2yz-2xyz)\mathbf{i}+x^2y\mathbf{j}+(yz^2-x^2z)\mathbf{k}$$

$$\text{Then Curl Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2yz-2xyz & x^2y & yz^2-x^2z \end{vmatrix} = z^2\mathrm{i}-(-2xz-2y+2xy)\mathbf{j}+(2xy-2z+2xz)\mathbf{k}$$

$\therefore$ At (2, 1, 1), curl cul $\mathbf{F}=\nabla\times(\nabla\times\mathbf{F}) = \mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$

Two interesting general results

(a) Curl grad ϕ where ϕ is a scalar

$$\text{grad } \phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

$$\therefore \text{curl grad } \phi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \mathbf{i}\left\{\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial x}\right\} - \mathbf{j}\left\{\frac{\partial^2\phi}{\partial z\partial x} - \frac{\partial^2\phi}{\partial x\partial z}\right\} + \mathbf{k}\left\{\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right\} = 0$$

$\therefore$ curl grad $\phi = \nabla\times(\nabla\phi) = 0$

(b) Div curl A where A is a vector.

$$\mathbf{A} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$$

$$\text{curl } \mathbf{A} = \nabla\times\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix} = \mathbf{i}\left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right) - \mathbf{j}\left(\frac{\partial a_z}{\partial x} - \frac{\partial a_x}{\partial z}\right) + \mathbf{k}\left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)$$

$$\text{Then div curl } \mathbf{A} = \nabla.(\nabla\times\mathbf{A}) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right).(\nabla\times\mathbf{A})$$

$$= \frac{\partial^2 a_z}{\partial x\partial y} - \frac{\partial^2 a_y}{\partial z\partial x} - \frac{\partial^2 a_z}{\partial x\partial y} + \frac{\partial^2 a_x}{\partial y\partial z} - \frac{\partial^2 a_y}{\partial z\partial x} - \frac{\partial^2 a_x}{\partial y\partial z} = 0$$

$\therefore$ div curl $\mathbf{A} = \nabla\,.\,(\nabla\times\mathbf{A}) = 0$

(c) Div grad ϕ where ϕ is a scalar.

$$\text{grad } \phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

$$\text{Then div grad } \phi = \nabla.(\nabla\phi) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right).\left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\right) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$\therefore \text{div grad } \phi = \nabla.(\nabla\phi) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

This result is sometimes denoted by $\nabla^2\phi$.

So these general results are

(a) curl grad $\phi = \nabla\times(\nabla\phi) = 0$
(b) div curl $\mathbf{A} = \nabla.(\nabla\times\mathbf{A}) = 0$
(c) div grad $\phi=\nabla.(\nabla\phi)=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}$