Lecture No -33 Examples

Example

Evaluate I = $\int_C$ {3ydx + (3x+2y)dy} from A(1, 2) to B (3, 5).

No path is given, so the integrand is doubtless an exact differential of some function z = f (x,y). In fact $\frac{\partial P}{\partial y} = 3 = \frac{\partial Q}{\partial x}$. We have already dealt with the integration of exact differentials, so there is no difficulty. Compare with

$I = \int_C \{P\,dx + Q\,dy\}$.

$P = \frac{\partial z}{\partial x} = 3y \qquad \therefore\ z = \int 3ydz = 3xy + f(y)$ -------- (i)

$Q = \frac{\partial z}{\partial y} = 3x + 2y \qquad \therefore\ z = \int (3x+2y)\,dy = 3xy + y^2 + F(x)$ (ii)

For (i) and (ii) to agree $f(y) = y^2$; $F(x) = 0$

Hence z = 3xy + y2

$\therefore I = \int_C \{3ydx + (3x+2y)dy\} = \int_{(1,2)}^{(3,5)} d(3xy+y^2) = \left[3xy+y^2\right]_{(1,2)}^{(3,5)} = (45+25) - (6+4) = 60$

Example

Evaluate I = $\int_C \{(x^2+ye^x)dx+(e^x+y)dy\}$ between A (0, 1) and B (1, 2).

As before, compare with $\int_C$ {Pdx+Q dy}.

$P = \frac{\partial z}{\partial x} = x^2 + ye^x \qquad \therefore\ z = \frac{x^3}{3} + ye^x + f(y)$

$Q = \frac{\partial z}{\partial y} = e^x + y \qquad \therefore\ z = ye^x + \frac{y^2}{2} + F(x)$

For these expressions to agree,

$f(y) = \frac{y^2}{2}$; $F(x) = \frac{x^3}{3}$ Then $I = \left[\frac{x^3}{3} + ye^x + \frac{y^2}{2}\right]_{(0,1)}^{(1,2)} = \frac{5}{6} + 2e$

So the main points are that, if (Pdx+Qdy) is an exact differential

(a) $I = \int_C (Pdx + Qdy)$ is independent of the path of integration

(b) $I = \oint_C (P\,dx + Q\,dy)$ is zero.

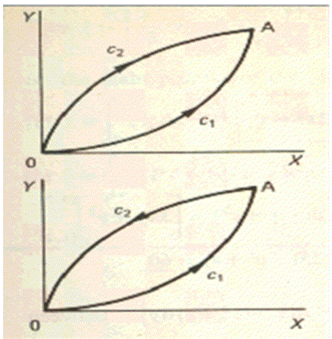
If $I = \int_C \{P\,dx + Q\,dy\}$ and (Pdx + Qdy) is an exact differential,

Then $I_{c_1} = I_{c_2}$

$I_{c_1} + I_{c_2} = 0$

Hence, the integration taken round a closed curve is zero, provided (Pdx+Q dy) is an exact differential.

$\therefore$ If (P dx + Q dy) is an exact differential, $\oint (P\,dx + Q\,dy) = 0$



Exact differentials in three independent variables

A line integral in space naturally involves three independent variables, but the method is very much like that for two independent variables.

$dz = Pdx + Q\,dy + R\,dw$ is an exact differential of $z = f(x, y, w)$

$$\text{if } \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x};\ \frac{\partial P}{\partial w} = \frac{\partial R}{\partial x};\ \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial w}$$

If the test is successful, then

(a) $\int_C (P\,dx + Q\,dy + R\,dw)$ is independent of the path of integration.

(b) $\oint_C (P\,dx + Q\,dy + R\,dw)$ is zero.

Example

Verify that $dz=(3x^2yw+6x)dx+(x^3w-8y)dy+(x^3y+1)\,dw$ is inexact differential and hence evaluate $\int_C dz$ from A (1, 2, 4) to B (2, 1 3).

First check that dz is an exact differential by finding the partial derivatives above, when

$P = 3x^2yw + 6x$; $Q = x^3w - 8y$; and $R = x^3y + 1$

$$\frac{\partial P}{\partial y} = 3x^2w;\ \frac{\partial Q}{\partial x} = 3x^2w\ \therefore \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$$\frac{\partial P}{\partial w} = 3x^2y;\ \frac{\partial R}{\partial x} = 3x^2y\ \therefore \frac{\partial P}{\partial w} = \frac{\partial R}{\partial x}$$

$$\frac{\partial R}{\partial y} = x^3;\ \frac{\partial Q}{\partial w} = x^3\ \therefore \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial w}$$

$\therefore$ dz is an exact differential

Now to find z. $P = \frac{\partial z}{\partial x}$; $Q = \frac{\partial z}{\partial y}$; $R = \frac{\partial z}{\partial w}$

$$\therefore \frac{\partial z}{\partial x} = 3x^2yw+6x\ \therefore\ z=\int(3x^2yw+6x)dx\ = x^3yw+3x^2+f(y)+F(w)$$

$$\therefore \frac{\partial z}{\partial y} = x^3w-8x\ \therefore\ z = \int(x^3w-8y)dy\ = x^3yw-4y^2+g(x)+F(w)$$

$$\frac{\partial z}{\partial w} = x^3y+1\ \ \therefore\ z = \int(x^3y+1)dw\ = y^3yw+w+f(y)+g(x)$$

For these three expressions for z to agree

$f(y) = -4y^2$; $F(w) = w$; $g(x) = 3x^2$

$$\therefore\quad z = x^3yw + 3x^2 - 4y^2 + w$$

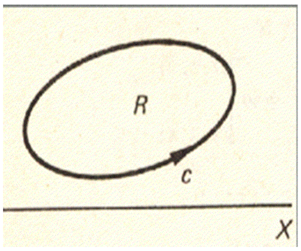
$$\therefore\quad I = \Big[x^3yw + 3x^2 - 4y^2 + w\Big]_{(1,2,4)}^{(2,1,3)}$$

$$\text{for}\quad I = \Big[x^3yw + 3x^2 - 4y^2 + w\Big]_{(1,2,4)}^{(2,1,3)} = (24+12-4+3)-(8+3-16+4)=36$$

The extension to line integrals in space is thus quite straightforward.

Finally, we have a theorem that can be very helpful on occasions and which links up with the work we have been doing. It is important, so let us start a new section.

Green's Thorem



Let P and Q be two function of x and y that are finite and continuous inside and the boundary c of a region R in the xy-plane.If the first partial derivatives are continuous within the region and on the boundary, then Green's theorem states that.

$$\iint_R \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dx\,dy = -\oint_C (P\,dx + Q\,dy)$$

That is, a double integral over the plane region R can be transformed into a line integral over the boundary c of the region – and the action is reversible.
Let us see how it works.

EXAMPLE

Evaluate $I = \oint_C \{(2x - y)dx + (2y+x)dy\}$ around the boundary c . the ellipse $x^2 + 9y^2 = 16$.
The integral is of the form

$$I = \oint_C \{P\,dx + Q\,dy) \text{ where } P = 2x - y \;\therefore\; \frac{\partial P}{\partial y} = -1 \text{ and } Q = 2y + x \;\therefore\; \frac{\partial Q}{\partial x} = 1.$$

$$\therefore\; I = -\iint_R \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dxdy = -\iint_R (-1-1)dx\,dy = 2\iint_R dx\,dy$$

But $\iint_R dx\,dy$ over any closed region give the area of the figure.
In this case, then, I = 24 where A is the area of the ellipse

$$x^2+9y^2 = 16 \text{ i.e. } \frac{x^2}{16} + \frac{9y^2}{16} = 1$$

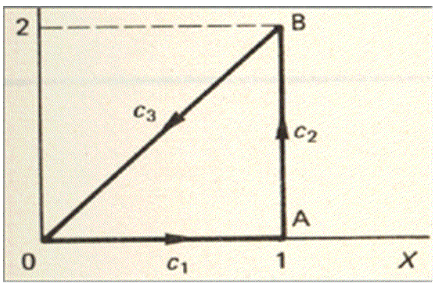
$$\therefore\; a = 4;\; b = \frac{4}{3} \qquad \therefore\; A = \frac{16\pi}{3} \qquad \therefore\; I = 2A = \frac{32\pi}{3}$$

To demonstrate the advantage of Green's theorem, let us work through the next example (a) by the previous method, and (b) by applying Green's theorem.

Example

Evaluate $I = \oint_C \{(2x+y)\,dx + (3x-2y)\,dy\}$ taken in anticlockwise manner round the triangle with vertices at O (0,0) A (1, 0) B (1, 2).

$$I = \oint_C \{(2x + y)\,dx + (3x - 2y)\,dy\}$$



(a) By the previous method

There are clearly three stages with c_1, c_2, c_3. Work through the complete evaluation to determine the value of I. It will be good revision. When you have finished, check the result with the solution in the next frame. I = 2

(a) (i) c_1 is $y = 0 \quad \therefore \quad dy = 0$

$$\therefore \quad I_1 = \int_0^1 2x\,dx = \left[x^2\right]_0^1 = 1 \quad \therefore \ I_1 = 1$$

(ii) c_2 is $x = 1 \quad \therefore \quad dx = 0$

$$\therefore \quad I_2 = \int_0^2 (3-2y)\,dy = \left[3y - y^2\right]_1^0 = 2 \quad \therefore \ I_2 = 2$$

(iii) c_3 is $y = 2x \quad \therefore \quad dy = 2\,dx$

$$\therefore \ I_3 = \int_1^0 \{4x\,dx + (3x - 4x)\,2\,dx\}$$

$$= \int_1^0 2x\,dx = \left[x^2\right]_1^0 = -1 \quad \therefore \ I_3 = -1$$

$$I = I_1 + I_2 + I_3 = 1 + 2 + (-1) = 2 \quad \therefore \ I = 2$$

Now we will do the same problem by applying Green's theorem, so more

(b) By Green's theorem

$$I = \oint_C \{(2x + y)\,dx + (3x - 2y)\,dy\}$$

$$P = 2x + y \quad \therefore \quad \frac{\partial P}{\partial y} = 1;$$

$$Q = 3x - 2y \quad \therefore \quad \frac{\partial Q}{\partial x} = 3$$

$$I = -\iint_R \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dx\,dy$$

Finish it off. I = 2

For $I = -\iint_R (1-3)\,dx\,dy = 2\iint_R dx\,dy = 2A$

$= 2 \times$ the area of the triangle $= 2 \times 1 = 2$

$\therefore \quad I = 2$

Application of Green's theorem is not always the quickest method. It is useful, however, to have both methods available.

If you have not already done so, make a note of Green's theorem.

$$\iint_R \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dx\,dy = -\oint_C (P\,dx + Q\,dy)$$

Note: Green's theorem can, in fact, be applied to a region that is not simply connected by arranging a link between outer and inner boundaries, provided the path of integration is such that the region is kept on the left-hand side.