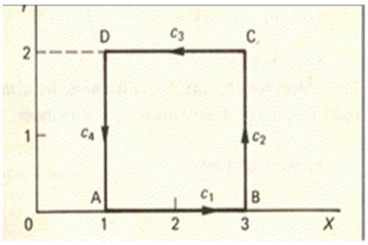
Lecture No -32 Examples

Example

Evaluate $I = \oint \{xydx+(1+y^2)dy\}$ where c is the boundary of the rectangle joining A(1,0), B (3, 0), C(3, 2), D (1, 2).



First draw the diagram and insert c_1, c_2, c_3, c_4.

That give

Now evaluate I_1 for AB; I_2 for BC; I_3 for CD; I_4 for DA; and finally I.

$I_1 = 0$; $I_2 = 4\frac{2}{3}$; $I_3 = -8$ $I_4 = -4\frac{2}{3}$; $I = -8$

Here is the complete working.

$$I = \oint \{xydx + (1 + y^2)\, dy\}$$

(a) AB: c_1 is $y = 0$ $\therefore$ $dy = 0$ $\therefore$ $I_1 = 0$

(b) BC: c_2 is $x = 3$ $\therefore$ $dx = 0$

$$\therefore I_2 = \int_0^2 (1+y^2)dy = \left[y + \frac{y^3}{3}\right]_0^2 = 4\frac{2}{3} \qquad \therefore I_2 = 4\frac{2}{3}$$

(c) CD : c_3 is $y = 2$ $\therefore$ $dy = 0$

$$\therefore I_3 = \int_3^1 2xdx = \left[x^2\right]_3^1 = -8 \quad \therefore I_3 = -8$$

(d) DA : c_4 is $x = 1$ $\therefore$ $dx = 0$

$$\therefore I_4 = \int_2^0 (1+y^2)\, dy = \left[y + \frac{y^3}{3}\right]_2^0 = -4\frac{2}{3}$$

Finally $I = I_1 + I_2 + I_3 + I_4 = 0 + 4\frac{2}{3} - 8 - 4\frac{2}{3} = -8$ $\therefore I = -8$

Remember that, unless we are directed otherwise, we always proceed round the closed boundary in an anticlockwise manner.

Line integral with respect to arc length

We have already established that

$$I = \int_{AB} F_t ds = \int_{AB} \{Pdx+Qdy\}$$

where F_t denoted the tangential force along the curve c at the sample point K(x,y).

The same kind of integral can, of course, relate to any function f(x,y) which is a function of the position of a point on the stated curve, so that

$$I = \int_C f(x, y)\, ds.$$

This can readily be converted into an integral in terms of x:

$$I = \int_C f(x,y)dx = \int_C f(x,y) \frac{ds}{dx} dx$$

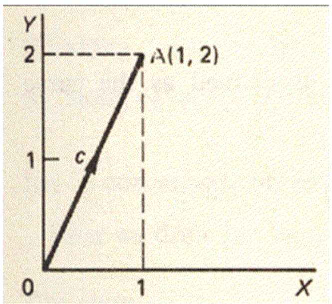
where $\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

$$\therefore \int_C f(x,y)\, dx = \int_{x_1}^{x_2} f(x,y)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \text{------------------} \quad (1)$$

Example

Evalaute I = $\int_C (4x+3xy)ds$ where c is the straight line joining O(0,0) to A (1,2).

c is the line y = 2x $\therefore$ $\frac{dy}{dx}$ = 2



$\therefore \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{5}$

$\therefore \; I = \int_{x=0}^{x=1} (4x+3xy)ds = \int_0^1 (4x+3xy)(\sqrt{5})\, dx$. But y = 2x

for $\quad I = \int_0^1 (4x+6x^2)(\sqrt{5})\, dx = 2\sqrt{5}\int_0^1 (2x+3x^2)\, dx = 4\sqrt{5}$

Parametric Equations

When x and y are expressed in parametric form, e.g. x = y (t), y = g(t), then

$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \qquad \therefore \; ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$I = \int_C f(x,y)ds = \int_{t_1}^{t_2} f(x,y)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ ----------------(2)

Example

Evaluate I = $\oint$ 4xyds where c is defined as the curve x = sin t, y = cos t between t=0 and t=$\frac{\pi}{4}$.

We have x = sin t $\quad \therefore \; \frac{dx}{dt}$ = cos t , y = cos t $\quad \therefore \; \frac{dy}{dt}$ = – sin t

$\therefore \qquad \frac{ds}{dt} = I$

for $\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\cos^2 t + \sin^2 t} = 1$

$\therefore \; I = \int_{t_1}^{t_2} f(x,y)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$= \int_0^{\pi/4} 4 \sin t \cos t \, dt = 2\int_0^{\pi/4} \sin 2t \, dt$

$= -2\left[\frac{\cos 2t}{2}\right]_0^{\pi/4} = 1$

Dependence of the line integral on the path of integration

We know that integration along two separate paths joining the same two end points does not necessarily give identical results.With this in mind, let us investigate the following problem.

EXAMPLE

Evaluate $I = \oint_C \{3x^2 y^2 dx + 2x^3 y\, dy\}$ between O (0, 0) and A (2, 4)

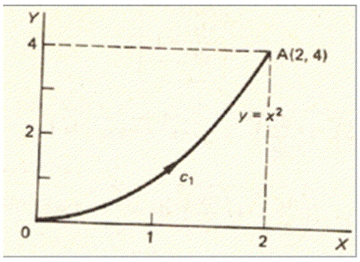
(a) along c_1 i.e. $y = x^2$

(b) along c_2 i.e. $y = 2x$

(c) along c_3 i.e. $x = 0$ from (0,0) to (0,4) and $y = 4$ from (0,4) to (2,4).

(a).First we draw the figure and insert relevant information.

$I = \int_C \{3x^2y^2dx + 2x^3ydy\}$



The path c_1 is $y = x^2$ $\therefore$ $dy = 2x\,dx$

$\therefore$ $I_1 = \int_0^2 \{3x^2x^4dx+2x^3x^22xdx\}= \int_0^2 (3x^6 + 4x^6)\,dx$

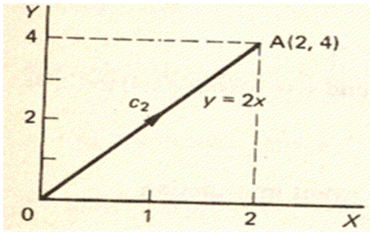
$\therefore$ $= \left[x^7\right]_0^2 = 128$ $\therefore$ $I_1 = 128$

(b) In (b), the path of integration changes to c_2, i.e. $y = 2x$

So, in this case,for with c_2, $y = 2x$ $\therefore$ $dy = 2dx$

$\therefore$ $I_2 = \int_0^2 (3x^2\, 4x^2\, dx + 2x^3\, 2x^2\, dx\}$

$= \int_0^2 20\, x^4\, dx= 4\left[x^5\right]_0^2 = 128$ $\therefore$ $I_2 = 128$



(c) In the third case, the path c_3 is split

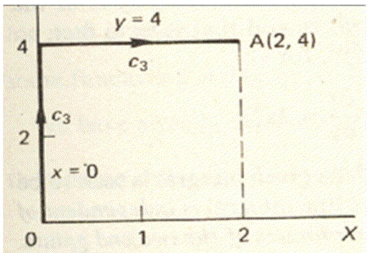
$x = 0$ from (0,0) to (0, 4), $y = 4$ from (0, 4) to (2, 4)

Sketch the diagram and determine I_3.

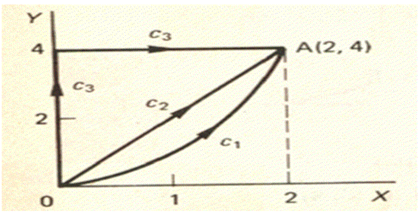
from (0,0) to (0,4) $x=0$ $\therefore$ $dx=0$ $\therefore$ $I_{3a}=0$

from (0,4) to (2,4) $y=4$ $\therefore$ $dy=0$ $\therefore$ $I_{3b}=48$

$\int_0^2 48x^2\, dx = 128$ $\therefore$ $I_3 = 128$



In the example we have just worked through, we took three different paths and in each case, the line integral produced the same result. It appears, therefore, that in this case, the value of the integral is independent of the path of integration taken.



We have been dealing with $I = \int_C \{3x^2y^2dx+2x^3ydy\}$

On reflection, we see that the integrand $3x^2 y^2 dx + 2x^3y dy$ is of the form Pdx+Qdy which we have met before and that it is, in fact, an exact differential of the function

$z = x^3y^2$, for $\frac{\partial z}{\partial x} = 3x^2 y^2$ and $\frac{\partial z}{\partial y} = 2x^3 y$

This always happens. If the integrand of the given integral is seen to be an exact differential, then the value o the line integral is independent of the path taken and depends only on the coordinates of the two end points.