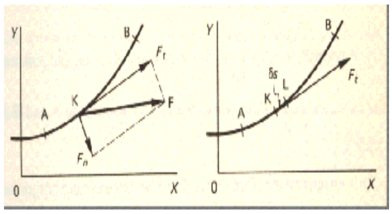
Lecture No -31 Line Integral



The work done in moving the particle through a small distance δs from K to L along the curve is then approximately $F_1\ \delta s$. So the total work done in moving a particle along the curve from A to B is given by

$\lim_{\delta \to 0} \sum F_t\ \delta s = \int F_t\ ds$ from A to B

This is normally written $\int_{AB} F_t\ ds$ where A and B are the end points of the curve, or as $\int_C F_t\ ds$ where the curve c connecting A and B is defined.Such an integral thus formed, is called a line integral since integration is carried out along the path of the particular curve c joining A and B.

$\therefore\ I = \int_{AB} F_t\ dx = \int_C F_t\ ds$

where c is the curve y = f(x) between $A(x_1, y_2)$ and $B(x_2, y_2)$.

There is in fact an alternative form of the integral which is often useful, so let us also consider that.

Alternative form of a line integral

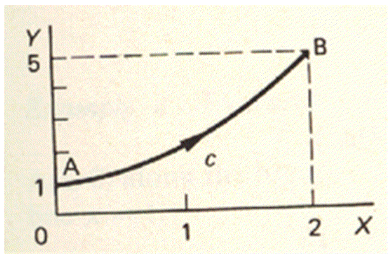
It is often more convenient to integrate with respect to x or y than to take arc length as the variable.

If F_t has a component ,P in the x-direction ,Q in the y-direction

then the work done from K to L can be stated as $P\delta x + Q\delta y$

Example 1

Evaluate $\int_C (x + 3y)\ dx$ from A (0, 1) to B (2, 5) along the curve $y = 1 + x^2$.



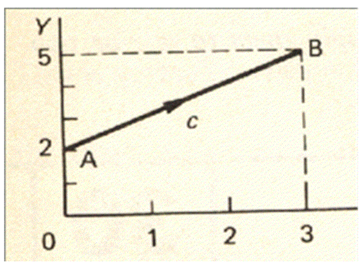
The line integral is of the form

$\int_C (P\ dx + Q dy)$ where, in this case, Q = 0 and c is the curve $y = 1 + x^2$.

It can be converted at once into an ordinary integral by substituting for y and applying the appropriate limits of x.

$I = \int_C (Pdx+Qdy) = \int_C (x+3y)dx = \int_C^2 (x+3+3x^2)dx$

$= \left[\frac{x^2}{2}+3x+x^3\right]_0^2 = 16$

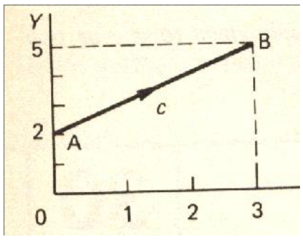


Now for another, so turn on.

Example 2

Evaluate $I = \int_C (x^2 + y)\,dx + (x - y^2)dy$ from A (0, 2) to B (2, 5) along the curve y = 2 + x.

$I = \int_C (Pdx + Qdy)$



$P = x^2 + y = x^2 + 2 + x = x^2 + x + 2$

$Q = x - y^2 = x - (4+4x+x^2) = -(x^2+3x+4)$

Also y = 2 + x

$\therefore$ dy = dx and the limits are x=0 to x=3

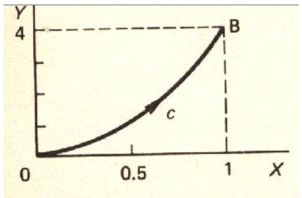
$\therefore$ I = – 15

for $I = \int_0^2 \{(x^2+x+2)\,dx - (x^2+3x+4)\,dx\}$ or $\int_0^2 -(2x+2)\,dx = \left[-x^2 - 2x\right]_0^2 = -4 - 4 = -8$

Example 3

Evaluate $I = \int_C \{(x^2+2y)dx + xydy\}$ from O(0, 0) to B(1, 4) along the curve $y=4x^2$.

In this case, c is the curve y = 4x2.



$\therefore$ dy = 8x dx

Substitute for y in the integral and apply the limits.

Then I = 9.4

for $\quad I = \int_C \{(x^2+2y)\,dx+xydy\}$

$y = 4x^2 \qquad \therefore\ dy = 8xdx$

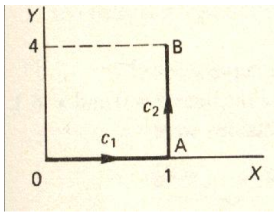
also $x^2 + 2y = x^2 + 8x^2 = 9x^2$; $\quad xy = 4x^3$

$\therefore\ I=\int_0^1 \{9x^2dx+32x^4dx\}=\int_0^1 (9x^2+32x^4)\,dx = 9.4$

They are all done in very much the same way.

Example 4

Evaluate $I = \int_C \{(x^2 + 2y)\,dx + xydy\}$ from O(0, 0) to A (1, 0) along the line y = 0 and then from A (1, 0) to B (1, 4) along the line x = 1.



(i) OA : c1 is the line y = 0 $\quad\therefore$ dy = 0.

Substituting y = 0 and dy = 0 in the given integral gives.

$$I_{OA} = \int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

(ii) AB: Here c_2 is the line x = 1 $\therefore$ dx=0

$\therefore\ I_{AB} = 8$

For $I_{AB} = \int_0^4 \{(1 + 2y)(0) + ydy\} = \int_0^4 ydy = \left[\frac{y^2}{2}\right]_0^4 = 8$

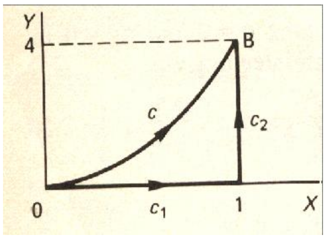
Then $I = I_{OA}+I_{AB} = \frac{1}{3} + 8 = 8\frac{1}{3} \qquad \therefore\ I=8\frac{1}{3}$

If we now look back to Example 3 and 4 just completed, we find that we have evaluated the same integral between the same two end points, but along different paths of integration.If we combine the two diagrams, we have

where c is the curve $y = 4x^2$ and $c1 + c^2$ are the lines

y = 0 and x = 1. The result obtained were

$I_c = 9\frac{2}{3}$ and $I_{c_1+c_2} = 8\frac{1}{3}$



Notice therefore that integration along two distinct paths joining the same two end points does not necessarily give the same results.

Properties of line integrals

1. $\int_C F ds = \int_C \{Pdx + Qdy\}$

2. $\int_{AB} F ds = -\int_{BA} F ds$ and $\int_{AB} \{Pdx+Qdy\} = \int_{BA} \{Pdx+Qdy\}$

i.e. the sign of a line integral is reversed when the direction of the integration along the path is reversed.

3. (a) For a path of integration parallel to the y-axis, i.e. x = k, dx = 0

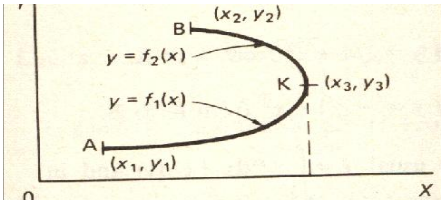
$\therefore \int_C Pdx = 0 \quad \therefore \; I_C = \int_C Qdy.$

(b) For a path of integration parallel to the x-axis, i.e. y = k, dy = 0.

$\therefore \int_C Qdy=0 \; \therefore \; I_C=\int_C Pdx.$

4. If the path of integration c joining A to B is divided into two parts AK and KB, then

$I_c = I_{AB} = I_{AK} + I_{KB}.$



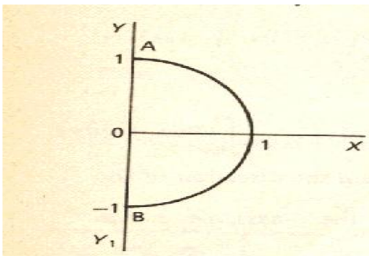
5. If the path of integration c is not single valued for part of its extent, the path is divided into two sections.

$y = f_1(x)$ from A to K, $y = f_2(x)$ from K to B.

6. In all cases, the actual path of integration involved must be continuous and single-valued.

Example

Evaluate $I = \int_C (x + y)\, dx$ from A(0, 1) to B (0, – 1) along the semi-circle $x^2+y^2=1$ for $x \geq 0$. The first thing we notice is that the path of integration c is not single-valued

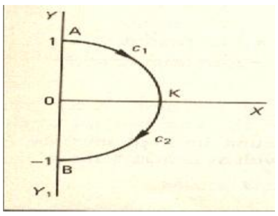


For any value of x, $y = \pm\sqrt{1 - x^2}$. Therefore, we divided c into two parts

(i) $y = \sqrt{1 - x^2}$ from A to K

(ii) $y = -\sqrt{1 - x^2}$ from K to B

As usual, $I = \int_C (Pdx + Qdy)$ and in this particular case, Q = 0



$$\therefore \; I = \int_C Pdx = \int_0^1 (x+\sqrt{1 - x^2})\, dx + \int_1^0 (x \sqrt{1 - x^2}\,)\, dx$$

$$= \int_0^1 (x + \sqrt{1 - x^2} - x+\sqrt{1 - x^2}\, dx \; = \; 2\int_0^1 \sqrt{1 - x^2}\, dx$$

Now substitute $x = \sin\theta$ and finish it off. $I = \frac{\pi}{2}$

for $I = 2\int_0^1 \sqrt{1 - x^2}dx \qquad x = \sin\theta$

$\therefore \quad dx = \cos\theta\, d\theta \quad \sqrt{1 - x^2} = \cos\theta$

Limits : $x = 0, \quad \theta = 0; \; x = 1, \; \theta = \frac{\pi}{2}$

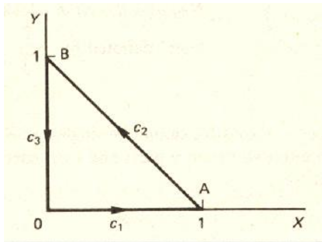
$\therefore \; I = 2\int_0^{\pi/2} \cos 2\theta d\theta = \int_0^{\pi/2} (1+\cos 2\theta)d\theta = \left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = \frac{\pi}{2}$

Now let us extend this line of development a stage further.

Example

Evaluate the line integral

$I = \oint (x^2dx - 2xy\, dy)$ where c comparises the three sides of the triangle joining O(0, 0), A (1, 0) and B (0, 1).



First draw the diagram and mark in c_1, c_2 and c_3, the proposed directions of integration. Do just that.The three sections of the path of integration must be arranged in an anticlockwise manner round the figure.

Now we deal with each pat separately.

(a) OA : c_1 is the line $y = 0$

Therefore, $dy = 0$.

Then $I = \oint (x^2dx - 2xy\, dy)$ for this part becomes

$I_1 = \int_0^1 x^2dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$ therefore $I_1 = \frac{1}{3}$

(b) AB : for c_2 is the line $y = 1 - x$

$\therefore \; dy = - dx.$

$I_2 = \int_0^1 \{x^2dx+2x(1-x)dx\} = \int_0^1 (x^2+2x-2x^2)dx = \int_0^1 (2x-x^2)\, dx = \left[x^2 - \frac{x^3}{3}\right]_0^1 = -\frac{2}{3}$

$\therefore \qquad I_2 = -\frac{2}{3}$

Note that anticlockwise progression is obtained by arranging the limits in the appropriate order.

Now we have to determine I_3 for BO.

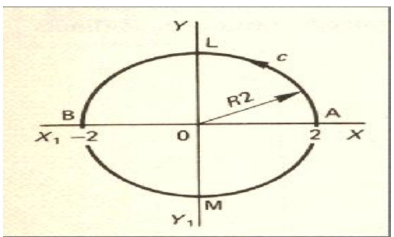
(c) BO: c_3 is the line $x = 0$

$\therefore \; dx = 0 \qquad \therefore \; I_3 = \int 0dy = 0 \quad \therefore \quad I_3 = 0$

Finally, $I = I_1+I_2+I_3 = \frac{1}{3} - \frac{2}{3} + 0 = -\frac{1}{3} \qquad \therefore \quad I = -\frac{1}{3}$

Example

Evaluate $\oint_c y\, dx$ when c is the circle $x^2+y^2 = 4$.



$x^2 + y^2 = 4 \qquad \therefore \qquad y = \pm\sqrt{4 - x^2}$

y is thus not single-valued. Therefore use

$y = \sqrt{4 - x^2}$ for ALB between

$x = 2$ and $x = -2$ and

$y = -\sqrt{4 - x^2}$ for BMA between

$x = -2$ and $x = 2$.

$$\therefore \quad I = \int_{2}^{-2} \sqrt{4-x^2}\,dx + \int_{-2}^{2} \{-\sqrt{4-x^2}\}\,dx$$

$$= 2\int_{2}^{-2} \sqrt{4-x^2}\,dx = -2\int_{-2}^{2} \sqrt{4-x^2}\,dx \qquad = -4\int_{0}^{2} \sqrt{4-x^2}\,dx.$$

To evaluate this integral, substitute $x = 2 \sin \theta$ and finish it off. $I = -4\pi$