Lecture No -30 Exact Differential

If z = f (x, y), then $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$

The result can be extended to functions of more than two independent variables.

If z = f(x, y, w), $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy + \frac{\partial z}{\partial w} dw$

Make a note of these results in differential form as shown.

Exercise

Determine the differential dz for each of the following functions.

1. $z = x^2 + y^2$
2. $z = x^3 \sin 2y$
3. $z = (2x - 1) e^{3y}$
4. $z = x^2 + 2y^2 + 3w^2$
5. $z = x^3 y^2 w$.

Finish all five and then check the result.

1. $dz = 2 (x\,dx + y\,dy)$
2. $dz = x^2 (3 \sin 2y\, dx + 2x \cos 2y\, dy)$
3. $dz = e^{3y} \{2dx + (6x - 3)\, dy\}$
4. $dz = 2 (xdx + 2ydy + 3wdw)$
5. $dz = x^2 y (3ywdx + 2xwdy + xydw)$

<u>Exact Differential</u>

We have just established that if z=f(x, y)

$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$

We now work in reverse.

Any expression dz = Pdx + Qdy, where P and Q are functions of x and y, is an exact differential if it can be integrated to determine z.

$\therefore\ P = \frac{\partial z}{\partial x}$ and $Q = \frac{\partial z}{\partial y}$

Now $\frac{\partial P}{\partial y} = \frac{\partial^2 z}{\partial y \partial x}$ and $\frac{\partial Q}{\partial x} = \frac{\partial^2 z}{\partial x \partial y}$ and we know that $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$

Therefore, for dz to be an exact differential $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ and this is the test we apply.

<u>Example</u>

$dz = (3x^2 + 4y^2)\, dx + 8xy\, dy$.

If we compare the right-hand side with Pdx + Qdy, then

$P = 3x^2 + 4y^2 \quad \therefore \quad \frac{\partial P}{\partial y} = 8y$

$Q = 8xy \qquad \therefore \quad \frac{\partial Q}{\partial x} = 8y$

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \therefore$ dz is an exact differential

Similarly, we can test this one.

<u>Example</u>

$dz = (1 + 8xy)\, dx + 5x^2\, dy$.

From this we find dz is not an exact differential

for $dz = (1 + 8xy)\, dx + 5x^2\, dy$

$\therefore\ P = 1 + 8xy \quad \therefore\ \dfrac{\partial P}{\partial y} = 8x$

$Q = 5x^2 \qquad \therefore\ \dfrac{\partial Q}{\partial x} = 10x$

$\dfrac{\partial P}{\partial y} \neq \dfrac{\partial Q}{\partial x}$ $\therefore$ dz is not an exact differential

Exercise

Determine whether each of the following is an exact differential.

1. $dz = 4x^3y^3dx + 3x^4y^2\, dy$
2. $dz = (4x^3y+2xy^3)\, dx+(x^4+3x_2y^2)\, dy$
3. $dz=(15y^2e^{3x}+2xy^2)dx+(10ye^{3x}+x^2y)dy$
4. $dz=(3x^2e^{2y}-2y^2e^{3x})dx+(2x^3e^{2y}-2ye^{3x})dy$
5. $dz=(4y^3\cos4x+3x^2\cos2y)dx+(3y^2\sin4x-2x^3\sin 2y)\, dy$.

1. Yes 2. Yes 3. No 4. No 5. Yes

We have just tested whether certain expressions are, in fact, exact differentials−and we said previously that, by definition, an exact differential can be integrated. But how exactly do we go about it? The following examples will show.

<u>Integration Of Exact Differentials</u>

$dz = Pdx+Qdy$ where $P=\dfrac{\partial z}{\partial x}$ and $Q=\dfrac{\partial z}{\partial y}$

$\therefore\ z = \int Pdx$ and also $z = \int Qdy$

<u>Example</u>

$dz = (2xy + 6x)\, dx + (x2 + 2y3)\, dy$.

$P = \dfrac{\partial z}{\partial x} = 2xy + 6x \quad \therefore\ z=\int (2xy+6x)dx$

$\therefore\ z = x^2y + 3x^2 + f(y)$ where f(y) is an arbitrary function of y only, and is akin to the constant of integration in a normal integral.

Also

$Q = \dfrac{\partial z}{\partial y} = x^2 + 2y^3 \quad \therefore\ z = \int (x^2+2y^3)\, dy$

$\therefore\ z = x2y + \dfrac{y^4}{2} + F(x)$ where F(x) is an arbitrary function of x only.

$$z = x^2y + 3x^2 + f(y) \qquad \text{(i)}$$

and $$z = x^2y + \frac{y^4}{2} + F(x) \qquad \text{(ii)}$$

For these two expressions to represent the same function, then

f(y) in (i) must be $\dfrac{y^4}{2}$ already in (i) and F(x) in (ii) must be $3x^2$ already in (i)

$\therefore \quad z = x^2y + 3x^2 + \dfrac{y^4}{2}$

Example

Integrate $dz = (8e^{4x} + 2xy^2)\,dx + (4\cos 4y + 2x^2y)\,dy$.
Argue through the working in just the same way, from which we obtain
$z = 2e^{4x} + x^2y^2 + \sin 4y$

Here it is.

$dz = (8e^{4x} + 2xy^2)\,dx+(4\cos 4y+2x^2y)\,dy$

$P = \dfrac{\partial x}{\partial x} = 8e^{4x} + 2xy^2$

$\therefore\; z = \int (8e^{4x} + 2xy^2)\,dx$

$\therefore\; z = 2e^{4x} + x^2y^2 + f(y) \qquad \text{(i)}$

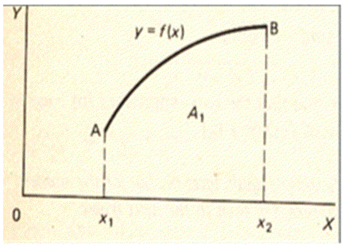
$Q = \dfrac{\partial z}{\partial y} = 4\cos 4y + 2x^2y$

$\therefore\; z = \int (4\cos 4y + 2x^2y)\,dy$

$\therefore\; z = \sin 4y + x^2y^2 + F(x) \qquad \text{(ii)}$

For (i) and (ii) to agree, $f(y) = \sin 4y$ and $F(x) = 2e^{4x}$

$\therefore\; z = 2e^{4x} + x^2y^2 + \sin 4y$



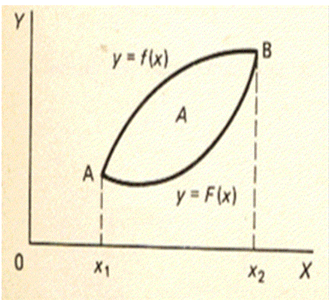
Area enclosed by the closed curve

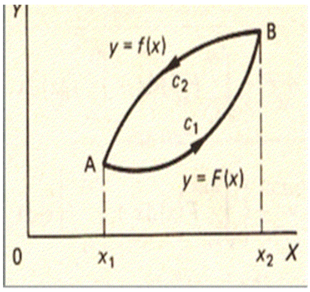
One of the earliest application of integration is finding the area of a plane figure bounded by the x-axis, the curve y = f (x) and ordinates at $x=x_1$ and $x=x_2$.

$A_1=\int_{x_1}^{x_2} y\,dx=\int_{x_1}^{x_2} f(x)\,dx$

If points A and B are joined by another curve, y = F(x)

$A_2 = \int_{x_1}^{x_2} f(x)\,dx$





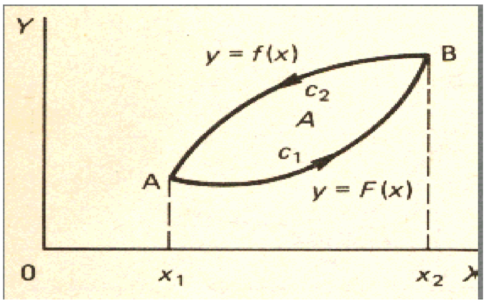
Combining the two figures, we have

$A=A_1-A_2 \qquad \therefore\; A=\int_{x_1}^{x_2} f(x)\,dx-\int_{x_1}^{x_2} f(x)\,dx$

The final result above can be written in the form

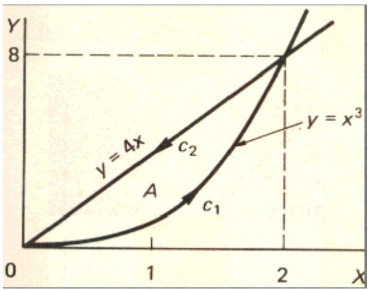
$$A = -\oint y\,dx$$

Where the symbol $\oint$ indicates that the integral is to be evaluated round the closed boundary in the positive



Example

Determine the area enclosed by the graph of $y = x^3$ and $y = 4x$ for $x \geq 0$.
First we need to know the points of intersection. These are $x = 0$ and $x = 2$
We integrate in a an anticlockwise manner
c_1: $y = x^3$, limits $x = 0$ to $x = 2$
c_2: $y = 4x$, limits $x = 2$ to $x = 0$.

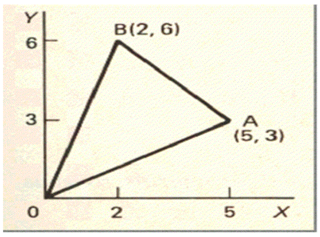


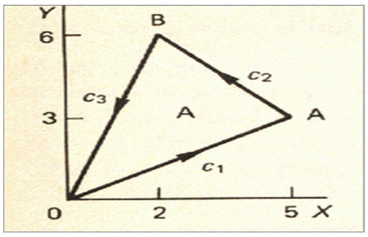
$A = -\oint y\,dx = A = 4$ square units

For $A = -\oint y\,dx = -\left\{\int_1^2 x^3 dx + \int_2^0 4x\,dx\right\} = -\left\{\left(\frac{x^4}{4}\right)_0^2 + \left[2x^2\right]_0^2\right\} = 4$

Example

Find the area of the triangle with vertices (0, 0), (5, 3) and (2, 6).





The equation of OA is $y = \frac{3}{5}x$, BA is $y = 8 - x$, OB is $y = 3x$

Then $A = -\oint y\,dx$

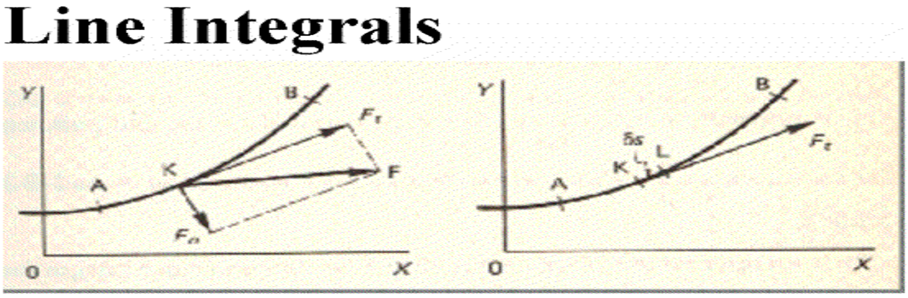
Write down the component integrals with appropriate limits.

$$A = -\oint y\,dx = -\left\{\int_0^5 \frac{3}{5}x\,dx + \int_5^2 (8-x)\,dx + \int_2^0 3x\,dx\right\}$$

The limits chosen must progress the integration round the boundary of the figure in an anticlockwise manner. Finishing off the integration, we have
A = 12 square units
The actual integration is easy enough. The work we have just done leads us on to consider line integrals, so let us make a fresh start in the next frame.

Line Integrals

If a field exists in the xy-plane, producing a force F on a particle at K, then F can be resolved into two components.F_1 along the tangent to the curve AB at K. Fa along the normal to the curve AB at K.



The work done in moving the particle through a small distance δs from K to L along the curve is then approximately F_1 δs. So the total work done in moving a particle along the curve from A to B is given by

$$\lim_{\delta \to 0} \sum F_t \, \delta s = \int F_t \, ds \text{ from A to B}$$

This is normally written $\int_{AB} F_t \, ds$ where A and B are the end points of the curve,

or as $\int_C F_t \, ds$ where the curve c connecting A and B is defined.

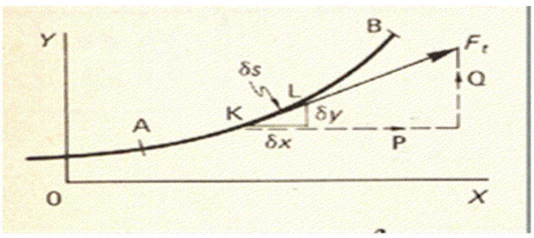
Such an integral thus formed, is called a line integral since integration is carried out along the path of the particular curve c joining A and B.

$$\therefore \quad I = \int_{AB} F_t \, dx = \int_C F_t \, ds$$

where c is the curve y = f(x) between A(x_1, y_2) and B (x_2, y_2).
There is in fact an alternative form of the integral which is often useful, so let us also consider that.

Alternative form of a line integral

It is often more convenient to integrate with respect to x or y than to take arc length as the variable.
If F_t has a component
P in the x-direction
Q in the y-direction
then the work done from K to L can be stated as Pδx + Qδy



$\therefore \int_{AB} F_t \, ds = \int_{AB} (P\,dx + Q dy)$

where P and Q are functions of x and y.

In general then, the line integral can be expressed as

$I = \int_C F_t \, ds = \int_C (P\,dx + Q dy)$

where c is the prescribed curve and F, or P and Q, are functions of x and y.

Make a note of these results –then we will apply them to one or two examples.