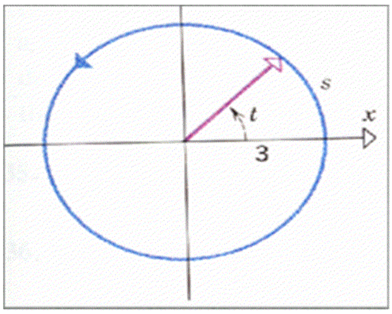
Lecture No -29 Change of parameter

It is possible for different vector-valued functions to have the same graph.
For example, the graph of the function
$r = (3 \cos t)\mathbf{i} + (3 \sin t)\mathbf{j},\ 0 \leq t \leq 2\pi$
is the circular of radius 3 centered at the origin with counterclockwise orientation. The parameter t can be interpreted geometrically as the positive angle in radians from the x-axis to the radius vector. For each value of t, let s be the length of the arc subtended by this angle on the circle



The parameters s and t are related by
$t = s/3, \quad 0 < s < 6\pi$
if we substitute this in (10), we obtain a vector-valued function of the parameter s, namely
$r = 3 \cos (s/3)\mathbf{i} + 3 \sin (s/3)\mathbf{j}, \quad 0 \leq s \leq 6\pi$
whose graph is also the circle of radius 3 centered at the origin with counterclockwise orientation .In various problems it is helpful to change the parameter in a vector-valued function by making an appropriate substitution. For example, we changed the parameter above from t to s by substituting $t=s/3$ in (10).
In general, if g is a real-valued function, then substituting $t = g(u)$ in r(t) changes the parameter from t to u.

When making such a change of parameter, it is important to ensure that the new vector-valued function of u is smooth if the original vector-valued function of t is smooth. It can be proved that this will be so if g satisfies the following conditions:

1. g is differentiable.
2. g' is continuous.
3. $g'(u) \neq 0$ for any u in the domain of g.
4. The range of g is the domain of **r**.

If g satisfies these conditions, then we call t = g(u) a smooth change of parameter. Henceforth, we shall assume that all changes of parameter are smooth, even if it is not stated explicitly.

ARC LENGTH

Because derivatives of vector-valued functions are calculated by differentiating components, it is natural do define integrals of vector-functions in terms of components.

EXAMPLE

If $x'(t)$ and $y'(t)$ are continuous for $a \le t \le b$, then the curve given by the parametric equations

$x = x(t),\ \ y = y(t)\ \ (a \le t \le b)$ (9)

has arc length

$$\mathbf{L} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad (10)$$

This result generalizes to curves in 3-spaces exactly as one would expect:

If $x'(t)$, $y'(t)$, and $z'(t)$ are continuous for $a \le t \le b$, then the curve given by the parametric equations

$x = x(t),\ \ y = y(t),\ \ z = z(t)\ \ (a \le t \le b)$

has arc length

$$\mathbf{L} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \qquad (12)$$

EXAMPLE

Find the arc length of that portion of the circular helix

$x = \cos t,\qquad y = \sin t,\qquad z = t$

From $t = 0$ to $t = \pi$

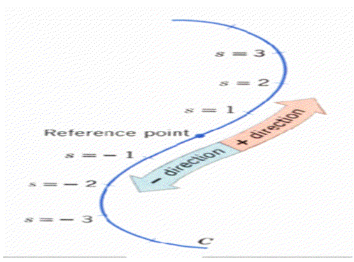
The arc length is

$$\mathbf{L} = \int_0^\pi \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \int_0^\pi \sqrt{(-\sin t)^2 + (\cos t)^2 + 1}\, dt$$

$$= \int_0^\pi \sqrt{2}\, dt = \sqrt{2}\,\pi$$

ARC LENTH AS A PARAMETER

For many purposes the best parameter to use for representing a curve in 2-space or 3-space parametrically is the length of arc measured along the curve from some fixed reference point. This can be done as follows:



Step 1: Select an arbitrary point on the curve C to serve as a reference point.

Step 2: Starting from the reference point, choose one direction along the curve to be the positive direction and the other to be the negative direction.

Step 3: If P is a point on the curve, let s be the “signed” arc length along C from the reference point to P, where s is positive if P is in the positive direction from the reference point, and s is negative if P is in the negative direction.

By this procedure, a unique point P on the curve is determined when a value for s is given. For example, s = 2 determines the point that is 2 units along the curve in the

positive direction from the reference point, and $s = -\frac{3}{2}$ determines the point that is $\frac{3}{2}$ units along the curve in the negative direction from the reference point.
Let us now treat s as a variable. As the value of s changes, the corresponding point P moves along C and the coordinates of P become functions of s. Thus, in 2-space the coordinates of P are (x(x,), y(s)), and in 3-space they are (x(s), y(s), z(s)). Therefore, in 2-space the curve C is given by the parametric equations x = x(s), y = y (s)
and in 3-space by x = x(s), y = y(s), z = z (s)

REMARKS

When defining the parameter s, the choice of positive and negative directions is arbitrary. However, it may be that the curve C is already specified in terms of some other parameter t, in which case we shall agree always to take the direction of increasing t as the positive direction for the parameter s. By so doing, s will increase as t increases and vice versa.
The following theorem gives a formula for computing an arc-length parameter s when the curve C is expressed in terms of some other parameter t. This result will be used when we want to change the parameterization for C from t to s.

THEOREM

(a) Let C be a curve in 2-space given parametrically by
x = x(t) , y = y (t)
where $x^{/}(t)$ and $y^{/}(t)$ are continuous functions. If an arc-length parameter s is introduced with its reference point at $(x(t_0), y(t_0))$, then the parameters s and t are related by

$$s = \int_{t_0}^{t} \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\, du \qquad (13a)$$

(b) Let C be a curve in 3-space given parametrically by
x = x(t), y = y(t), z = z(t)
where $x^{/}(t)$, $y^{/}(t)$, and $z^{/}(t)$ are continuous functions. If an arc-length parameter s is introduced with its reference point at $(x(t_0), y(t_0), z(t_0))$, then the parameters s and t are related by

$$s = \int_{t_0}^{t} \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du \qquad (13b)$$

Proof

If $t > t_0$, then from (10) (with *u* as the variable of integration rather than *t*) it follows that

$$\int_{t_0}^{t} \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\, du \qquad (14)$$

represents the arc length of that portion of the curve C that lies between $(x(t_0), y(t_0))$ and (x (t), y(t)). If $t < t_0$, then (14) is the negative of this arc length. In either case, integral (14) represents the "signed" arc length *s* between these points, which proves (13a).
It follows from Formulas (13a) and (13b) and the Second Fundamental Theorem of Calculus (Theorem 5.9.3) that in 2-space.

$$\frac{ds}{dt} = \frac{d}{dt}\left[\int_{t_0}^{t} \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\, du\right] = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

and in 3-space

$$\frac{ds}{dt} = \frac{d}{dt}\left[\int_{t_0}^{t}\sqrt{\left(\frac{dx}{du}\right)^2+\left(\frac{dy}{du}\right)^2+\left(\frac{dz}{du}\right)^2}\,dt\right] = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}$$

Thus, in 2-space and 3-space, respectively,

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2} \qquad (15a)$$

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2} \qquad (15b)$$

REMARKS:

Formulas (15a) and (15b) reveal two facts worth noting. First, ds/dt does not depend on t_0; that is, the value of ds/dt is independent of where the reference point for the parameter s is located. This is to be expected since changing the position of the reference point shifts each value of s by a constant (the arc length between the reference points), and this constant drops out when we differentiate. The second fact to be noted from (15a) and (15b) is that $ds/dt \geq 0$ for all t. This is also to be expected since s increases with t by the remark preceding Theorem 15.3.2. If the curve C is smooth, then it follows from (15a) and (15b) that $ds/dt \geq 0$ for all t .

EXAMPLE

$x = 2t + 1,\ y = 3t - 2$ (16)

using arc length s as a parameter, where the reference point for s is the point (1, – 2).
In formula (13a) we used u as the variable of integration because t was needed as a limit of integration. To apply (13a), we first rewrite the given parametric equations with u in place of t; this gives
from which we obtain

$x = 2u + 1, \quad y = 3u - 2$

$\frac{dx}{du} = 2, \qquad \frac{dy}{du} = 3$

we see that the reference point (1,–2) corresponds to $t = t_0 = 0$

$$s = \int_{t_0}^{t}\sqrt{\left(\frac{dx}{du}\right)^2+\left(\frac{dy}{du}\right)^2}\,du = \int_{t_0}^{t}\sqrt{13}\,du = \sqrt{13}u\,\Big]_{u=0}^{u=t} = \sqrt{13}t$$

Therefore, $t = \frac{1}{\sqrt{13}}s$

Substituting this expression in the given parametric equations yields.

$$x = 2\left(\frac{1}{\sqrt{13}}s\right) + 1 = \frac{2}{\sqrt{13}}s + 1$$

$$y = 3\left(\frac{1}{\sqrt{13}}s\right) - 2 = \frac{3}{\sqrt{13}}s - 2$$

EXAMPLE

Find parametric equations for the circle $x = a\cos t$, $y = a\sin t$ $(0 \leq t \leq 2\pi)$
using arc length s as a parameter, with the reference point for s being (a, 0), where a > 0.
We first replace t by u in the given equations so that $x = a\cos u$, $y = a\sin u$

And $\frac{dx}{du} = -a\sin u$, $\frac{dy}{du} = a\cos u$

Since the reference point (a, 0) corresponds to t = 0, we obtain

$$s = \int_{t_0}^{t} \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\, du = \int_{t_0}^{t} \sqrt{(-a\sin u)^2 + (a\cos u)^2}\, du = \int_0^t a\, du = au\Big]_{u=0}^{u=t} = at$$

Solving for t in terms of s yields $t = s/a$

Substituting this in the given parametric equations and using the fact that $s = at$ ranges from 0 to $2\pi a$ as t ranges from 0 to 2π, we obtain

$$x = a\cos(s/a),\quad y = a\sin(s/a)\ (0 \le s \le 2\pi a)$$

<u>Example</u>

Find Arc length of the curve $\mathbf{r}(t) = t^3\mathbf{i} + t\mathbf{j} + \frac{1}{2}\sqrt{6}\, t^2\,\mathbf{k}$, $1 \le t \le 3$

Here $x = t^3$, $y = t$, $z = \frac{1}{2}\sqrt{6}\, t^2$

$$\frac{dx}{dt} = 3t^2,\quad \frac{dy}{dt} = 1,\quad \frac{dz}{dt} = \sqrt{6}\, t$$

$$\text{Arc length} = \int_1^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \int_1^3 \sqrt{9t^4 + 1 + 6t^2}\, dt = \int_1^3 \sqrt{(3t^2+1)^2}\, dt$$

$$= \left|t^3 + t\right|_1^3 = (3)^3 + 3 - (1)^3 - 1 = 27 + 3 - 1 - 1 = 28$$

<u>EXAMPLE</u>

Calculate $\frac{d\mathbf{r}}{d\mathbf{u}}$ by chain Rule.

$\mathbf{r} = e^t\mathbf{i} + 4e^{-t}\mathbf{j}$

$$\frac{dr}{dt} = e^t\mathbf{i} - 4e^{-t}\mathbf{j}$$

$$\frac{dt}{du} = 2u$$

$$\frac{d\mathbf{r}}{du} = \frac{d\mathbf{r}}{dt} \cdot \frac{dt}{du} = (e^t\mathbf{i} - 4e^t\mathbf{j}).(2u) = 2u\, e^{u^2}\mathbf{i} - 8ue^{-u^2}\mathbf{j}$$

By expressing $\mathbf{r}$ in terms of u

$R = e^u\mathbf{i} + 4e^{-u^2}\mathbf{j}$

$$\frac{d\mathbf{r}}{du} = 2u\, e^{u^2}\mathbf{i} - 8ue^{-u^2}\mathbf{j}$$