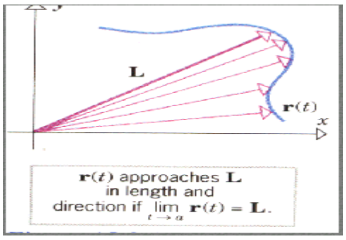
Lecture No -28 Limits of Vector Valued Functions

The limit of a vector-valued functions is defined to be the vector that results by taking the limit of each component. Thus, for a function r(t) = x (t)**i** + y (t)**j** in 2-space we define.

$$\lim_{t\to\alpha} r(t) = (\lim_{t\to\alpha} x(t))\mathbf{i} + (\lim_{t\to\alpha} y(t))\mathbf{j}$$

and for a function r(t) = x(t)**i** + y(t)**j** + z(t)**k**
in 3-space we define.

$$\lim_{t\to\alpha} r(t)=(\lim_{t\to\alpha} x(t))\mathbf{i}+(\lim_{t\to\alpha} y(t))\mathbf{j}+(\lim_{t\to\alpha} z(t))k$$



If the limit of any component does not exist, then we shall agree that the limit of r (t) does not exist. These definitions are also applicable to the one-sided and infinite limits $\lim_{t\to\alpha^+}$, $\lim_{t\to\alpha^-}$ $\lim_{t\to+\infty}$, and $\lim_{t\to-\infty}$. It follows from (1) and (2) that

$$\lim_{t\to\alpha} r(t) = L$$

if and only if the components of r(t) approach the components of L as t → a. Geometrically, this is equivalent to stating that the length and direction of r (t) approach the length and direction of L as t → α

CONTINUITY OF VECTOR-VALUED FUNCTIONS

The definition of continuity for vector-valued functions is similar to that for real-valued functions. We shall say that r is continuous at t_0 if

1. r (t_0) is defined;
2. $\lim_{t\to t_0} r(t)$ exists;
3. $\lim_{t\to t_0} r(t) = r(t_0)$.

It can be shown that r is continuous at t_0 if and only if each component of r is continuous. As with real-valued functions, we shall call r continuous everywhere or simply continuous if r is continuous at al real values of t. geometrically, the graph of a continuous vector-valued function is an unbroken curve.

DERIVATIVES OF VECOR-VALUED FUNCTIONS

The definition of a derivative for vector-valued functions is analogous to the definition for real-valued functions.

DEFINITION

The derivative $r^{/}(t)$ of a vector-valued function r(t) is defined by

$$\mathbf{r^{/}(t) = \lim_{h\to 0} \frac{r\,(t+h) - r\,(t)}{h}}$$

Provided this limit exists.

For computational purposes the following theorem is extremely useful; it states that the derivative of a vector-valued function can be computed by differentiating each components.

THEOREM

(a) If $r(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ is a vector-valued function in 2-space, and if x(t) and y(t) are differentiable, then

$$r'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j}$$

(b) If $r(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ is a vector-valued function in 3-space, and if x(t), y(t), and z(t) are differentiable, then

$$r'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$$

We shall prove part (a). The proof of (b) is similar.

Proof (a):

$$\mathbf{r}'(t) = \lim_{h\to 0}\frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \lim_{h\to 0}\frac{[x(t+h)-x(t)]}{h}\mathbf{i} + \lim_{h\to 0}\frac{[y(t+h)-y(t)]}{h}\mathbf{j}$$

$$= x'(t)\mathbf{i} + y'(t)\mathbf{j}$$

As with real-valued functions, there are various notations for the derivative of a vector-valued function. If r = r (t), then some possibilities are

$$\frac{d}{dt}[r(t)],\ \frac{dr}{dt},\ \mathbf{r}'(t), \text{ and } \mathbf{r}'$$

EXAMPLE

Let $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$. Find $\mathbf{r}'(t)$ and $\mathbf{r}'(1)$

$$\mathbf{r}'(t) = \frac{d}{dt}[t^2]\,\mathbf{i} + \frac{d}{dt}[t^3]\,\mathbf{j}$$

$$= 2t\,\mathbf{i}\ \ 3t^2\,\mathbf{j}$$

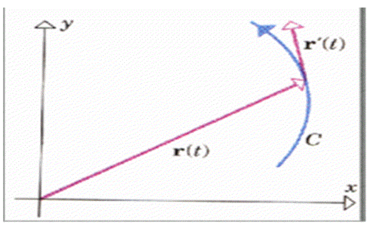
Substituting t=1 yields

$$\mathbf{r}'(1) = 2\mathbf{i}+3\mathbf{j}.$$

TAGENT VECTORS AND TANGENT LINES

GEOMETRIC INTERPRETATIONS OF THE DERIVATIVE.

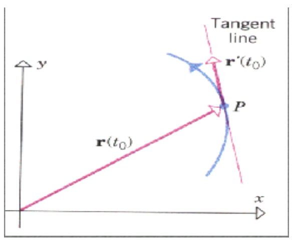
Suppose that C is the graph of a vector-valued function r(t) and that $\mathbf{r}'(t)$ exists and is nonzero for a given value of t. If the vector $\mathbf{r}'(t)$ is positioned with its initial point at the terminal point of the radius vector



DEFINITION

Let P be a point on the graph of a vector-valued function $\mathbf{r}(t)$, and let $\mathbf{r}(t_0)$ be the radius vector from the origin to P

If $\mathbf{r}'(t_0)$ exists and $\mathbf{r}'(t_0) \neq \mathbf{0}$, then we call $\mathbf{r}'(t_0)$ the tangent vector to the graph of $\mathbf{r}$ at $\mathbf{r}(t_0)$



REMARKS

Observe that the graph of a vector-valued function can fail to have a tangent vector at a point either because the derivative in (4) does not exist or because the derivative is zero at

the point.If a vector-valued function **r**(t) has a tangent vector $\mathbf{r}'(t_0)$ at a point onits graph, then the line that is parallel to $\mathbf{r}'(t_0)$ and passes through the tip of the radius vector $\mathbf{r}(t_0)$ is called the tangent line of the graph of **r**(t) at $\mathbf{r}(t_0)$
Vector equation of the tangent line is

$$\mathbf{r} = \mathbf{r}\,(t_0) + t\,\mathbf{r}'(t_0)$$

EXAMPLE

Find parametric equation of the tangent line to the circular helix
x = cost, y = sint, z = 1 at the point where t = π/6

To find a vector equation of the tangent line, then we shall equate components to obtain the parametric equations. A vector equation r=r(t) of the helix is

$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$

Thus, $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$

$\Rightarrow \mathbf{r}'(t) = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + k$

At the point where t = π/6, these vectors are

$$\mathbf{r}\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{\pi}{6}\mathbf{k} \quad \text{and}$$

$$\mathbf{r}'\left(\frac{\pi}{6}\right) = -\frac{1}{2}\,\mathrm{i} + \frac{\sqrt{3}}{2}\mathbf{j} + \mathbf{k}$$

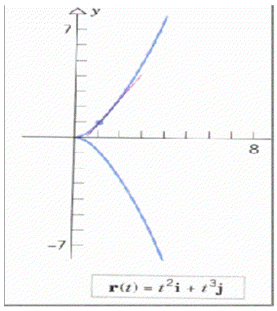
so from (5) with $t_0 = \pi/6$ a vector equation of the tangent line is

$$\mathbf{r}\left(\frac{\pi}{6}\right) + t\,\mathbf{r}'\left(\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{\pi}{6}\mathbf{k}\right) + t\left(-\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j} + \mathbf{k}\right)$$

Simplifying, then equating the resulting components with the corresponding components of $r = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ yields the parametric equation.

$$x = \frac{\sqrt{3}}{2} - \frac{1}{2}t,\quad y = \frac{1}{2} + \frac{\sqrt{3}}{2}t,\quad z = \frac{\pi}{6} + t$$

EXAMPLE



The graph of $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$ is called a semicubical parabola
Find a vector equation of the tangent line to the graph of **r**(t) at
(a) the point (0,0) (b) the point (1,1)
The derivative of **r**(t) is

$\mathbf{r}'(t) = 2t\mathbf{i} + 3t^2\mathbf{j}$

(a) The point (0, 0) on the graph of **r** corresponds to t = 0. As this point we have $\mathbf{r}'(0) = 0$, so there is no tangent vector at the point and consequently a tangent line does not exist at this point.

(b) The point (1, 1) on the graph of r corresponds to t = 1, so from (5) a vector equation of the tangent line at this point is $\mathbf{r} = \mathbf{r}(1) + t\,\mathbf{r}'(1)$

From the formulas for r (t) and $\mathbf{r}'(t)$ with t = 1, this equation becomes

$$\mathbf{r} = (\mathbf{i} + \mathbf{j}) + t\,(2\mathbf{i} + 3\mathbf{j})$$

If **r** is a vector-valued function in 2-space or 3-space, then we say that **r**(t) is smoothly parameterized or that **r** is a smooth function of t if the components of r have continuous derivatives with respect to t and $\mathbf{r}'(t) \neq \mathbf{0}$ for any value of t. Thus, in 3-space $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$

is a smooth function of t if $x'(t)$, $y'(t)$, and $z'(t)$ are continuous and there is no value of t at which al three derivatives are zero. A parametric curve C in 2-space or 3-space will be called smooth if it is the graph of some smooth vector-valued function.
It can be shown that a smooth vector-valued function has a tangent line at every point on its graph.

PROPERTIES OF DERIVATIVES

(Rules of Differentiation).

In either 2-space or 3-space let $\mathbf{r}(t)$, $\mathbf{r}_1(t)$, and $\mathbf{r}_2(t)$ be vector-valued functions, f(t) a real-valued function, k a scalar, and c a fixed (constant) vector. Then the following rules of differentiation hold:

$$\frac{d}{dt}[c] = 0$$

$$\frac{d}{dt}[kr(t)] = k\frac{d}{dt}[r(t)]$$

$$\frac{d}{dt}[\mathbf{r}_1(t) + \mathbf{r}_2(t)] = \frac{d}{dt}[\mathbf{r}_1(t)] + \frac{d}{dt}[\mathbf{r}_2(t)]$$

$$\frac{d}{dt}[\mathbf{r}_1(t) - \mathbf{r}_2(t)] = \frac{d}{dt}[\mathbf{r}_1(t)] - \frac{d}{dt}[\mathbf{r}_2(t)]$$

$$\frac{d}{dt}[f(t)\mathbf{r}(t)] = f(t)\frac{d}{dt}[\mathbf{r}(t)] + \mathbf{r}(t)\frac{d}{dt}[f(t)]$$

In addition to the rules listed in the foregoing theorem, we have the following rules for differentiating dot products in 2-space or 3-space and cross products in 3-space:

$$\frac{d}{dt}[\mathbf{r}_1(t)\cdot\mathbf{r}_2(t)] = \mathbf{r}_1\cdot\frac{dr_2}{dt} + \frac{dr_1}{dt}\cdot\mathbf{r}_2 \qquad (6)$$

$$\frac{d}{dt}[\mathbf{r}_1(t)\times\mathbf{r}_2(t)] = \mathbf{r}_1\times\frac{dr_2}{dt} + \frac{dr_1}{dt}\times\mathbf{r}_2 \qquad (7)$$

REMARK:

In (6), the order of the factors in each term on the right does not matter, but in (7) it does.
In plane geometry one learns that a tangent line to a circle is perpendicular to the radius at the point of tangency. Consequently, if a point moves along a circular arc in 2-space, one would expect the radius vector and the tangent vector at any point on the arc to be perpendicular. This is the motivation for the following useful theorem, which is applicable in both 2-space and 3-space.

THEOREM:

If r (t) is a vector-valued function in 2-space or 3-space and $\|\mathbf{r}(t)\|$ is constant for all t, then $\mathbf{r}(t)\cdot\mathbf{r}'(t) = 0$

that is, $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal vectors for all t.It follows from (6) with $\mathbf{r}_1(t)=\mathbf{r}_2(t)=\mathbf{r}(t)$ that

$$\frac{d}{dt}[\mathbf{r}(t)\cdot\mathbf{r}(t)] = \mathbf{r}(t)\cdot\frac{d\mathbf{r}}{dt} + \frac{d\mathbf{r}}{dt}\cdot\mathbf{r}(t)$$

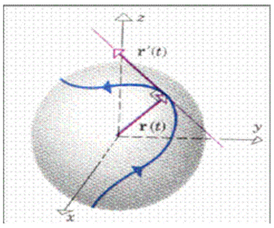
or, equivalently, $\frac{d}{dt}\,[\|\mathbf{r}(t)\|]^2 = 2\mathbf{r}(t)\,\boldsymbol{.}\,\frac{d\mathbf{r}}{dt}$

But $\|\mathbf{r}(t)\|^2$ is constant, so its derivative is zero. Thus $2r(t).\frac{d\mathbf{r}}{dt} = 0$ that is $\mathbf{r}(t)\,\boldsymbol{.}\,\frac{d\mathbf{r}}{dt} = 0$

That is the $\mathbf{r}(t)$ is perpendicular $\frac{d\mathbf{r}}{dt}$

EXAMPLE

Just as a tangent line to a circular arc in 2-space is perpendicular to the radius at the point of tangency, so a tangent line to a curve on the surface of a sphere in 3-space is perpendicular to the radius at the point of tangency.



To see that this is so, suppose that the graph of $\mathbf{r}(t)$ lies on the surface of the sphere of radius $k > 0$ centered at the origin.For each value of t we have $\|\mathbf{r}(t)\|=k$,

$$\mathbf{r}(t).\ \mathbf{r}'(t) = 0$$

which implies that the radius vector $\mathbf{r}(t)$ and the tangent vector $\mathbf{r}'(t)$ are perpendicular. This completes the argument because the tangent line, where it exists, is parallel to the tangent vector.

INTEGRALS OF VECTOR VALUED FUNCTION

(a) If r(t)=x(t)I +y(t) j is a vector-valued function in 2-space ,the we define.

$$\int r(t)dt = (\int x(t)dt)i + (\int y(t)dt)\,j \qquad (1a)$$

$$\int_a^b r(t)dt = (\int_a^b x(t)dt)i + (\int_a^b y(t)dt)\,j \qquad (1b)$$

(b) If r(t) = x (t) **i** + y(t) **j** + z(t)k is a vector-valued function in 3-space, then we define.

$$\int \mathbf{r}(t)dt = \left(\int x(t)dt\right)\mathbf{i} + \left(\int y(t)\,dt\right)\mathbf{j} + \left(\int z(t)dt\right)\mathbf{k} \qquad \textbf{(2a)}$$

$$\int_a^b r(t)dt = \left(\int_a^b x(t)dt\right)\mathbf{i} + \left(\int_a^b y(t)dt\right)\mathbf{j} + \left(\int_a^b z(t)dt\right)\mathbf{k} \qquad \textbf{(2b)}$$

$Let\ r(t) = 2ti + 3t^2 j$

$(a)\int r(t)dt \qquad (b)\int_0^2 r(t)dt$

$\int r(t)dt = \int (2ti + 3t^2 j)dt = (\int 2tdt)i + (\int\ 3t^2 dt)j$

$(t^2 + C_1)i + (t^3 + C_2)j = t^2 i + C_1 i + t^3 j + C_2 j = t^2 i + t^3 j + C_1 i + C_2 j = t^2 i + t^3 j + C$

$Where\ C = C_1 i + C_2 j\ is\ an\ arbitrary\ vector\ constant\ of\ integration$

$(b)\int_0^2 r(t)dt = \int_0^2 (2ti + 3t^2 j)dt = (\int_0^2 2tdt)i + (\int_0^2 3t^2 dt)j = \left[t^2\right]_0^2 i + \left[t^3\right]_0^2 j = (2^2 - 0)i + (2^3 - 0)j = 4i + 8j$

PROPERTEIS OF INTEGRALS

$\int cr(t)\ dt = c\int r\ (t)\ dt$ (3)

$\int [r_1(t)+r_2(t)]\ dt = \int r1(t)dt + \int r_2(t)\ dt$ (4)

$\int [r_1(t)-r_2(t)]dt = \int r_1(t)\ dt - \int r_2(t)\ dt$ (5)

These properties also hold for definite integrals of vector-valued functions. In addition, we leave it for the reader to show that if r is a vector-valued function in 2-space or 3-space, then $\frac{d}{dt}\left[\int r(t)\ dt\right] = r\ (t)$ (6)

This shows that an indefinite integrals of **r**(t) is, in fact, the set of antiderivatives of r(t), just as for real-valued functions.

If **r**(t) is any antiderivative or **r**(t) in the sense that $\mathbf{R}'(t) = \mathbf{r}(t)$, then

$\int r(t)\ dt = \mathbf{R}(t) + \mathbf{C}$ (7)

where C is an arbitrary vector constant of integration. Moreover,

$\int_a^b \mathbf{r}(t)\ dt = \mathbf{R}(t)\ \Big]_a^b = \mathbf{R}(b) - \mathbf{R}(a).$