Lecture No -27 Vector Valued Functions

Recall that a function is a rule that assigns to each element in its domain one and only one element in its range. Thus far, we have considered only functions for which the domain and range are sets of real numbers; such functions are called real-valued functions of a real variable or sometimes simply real-valued functions. In this section we shall consider functions for which the domain consists of real numbers and the range consists of vectors in 2-space or 3-space; such functions are called vector-valued functions of a real variable or more simply vector-valued functions. In 2-space such functions can be expressed in the form.

r (t) = (x (t), y (t)) = x (t) i + y (t) j

and in 3-space in the form

r (t) = (x (t), y (t), z (t))i+y(t)j+z (t)k

where x(t), y(t), and z(t) are real-valued functions of the real variable t. These real-valued functions are called the component functions or components of **r**. As a matter of notation, we shall denote vector-valued functions with boldface type [**f(t), g(t)**, and **r** (t) and real-valued functions, as usual, with lightface italic type [*f(t), g(t), and r(t)*].

EXAMPLE

$\mathbf{r}(t) = (\ln t)\,\mathbf{i} + \sqrt{t^2+2}\,\mathbf{j} + (\cos t\pi)\mathbf{k}$

then the component functions are $x(t)=\ln t, y(t)=\sqrt{t^2+2}$, and $z(t) = \cos t\pi$

The vector that r(t) associates with t = 1 is $r(1)=(\ln 1)\,\mathbf{i}+\sqrt{3}\mathbf{j} + (\cos \pi)\,\mathbf{k}=\sqrt{3}\mathbf{j} - \mathbf{k}$

The function **r** is undefined if $t \le 0$ because ln *t* is undefined for such *t*.

If the domain of a vector-valued function is not stated explicitly, then it is understood to consist of all real numbers for which every component is defined and yields a real value. This is called the natural domain of the function. Thus the natural domain of a vector-valued function is the intersection of the natural domains of its components.

PARAMETRIC EQUATIONS IN VECTOR FORM

Vector-valued functions can be used to express parametric equations in 2-space or 3-space in a compact form.

For example, consider the parametric equations x = x(t), y = y (t)

Because two vectors are equivalent if and only if their corresponding components are equal, this pair of equations can be replaced by the single vector equation.

x = x(t), y = y (t)

x **i** + y **j** = x (t)**i** + y (t) **j**

Similarly, in 3-space the three parametric equations

x = x(t), y = y (t), z = z (t)

can be replaced by the single vector equation

x**i** + y**j** + z**k** = x(t)**i** + y(t)**j** + z(t)**k**

if we let r = x **i** +y **j** and **r**(t) = x(t) **i** + y(t) **j** in 2-sapce
and let **r** = x**i** + y**j** + z**k** and **r**(t) = x(t) **i** + y(t)**j** + z(t)**k**
in 3-space, then both (2) and (4) can be written as **r** = **r**(t)
which is the vector form of the parametric equations in (1) and (3). Conversely, every vector equation of form (5) can be rewritten as parametric equations by equating components on the two sides.

EXAMPLE

Express the given parametric equations as a single vector equation.

(a) $x = t^2$, $y = 3t$
(b) x = cost, y = sint, z = t
(a) Using the two sides of the equations as components of a vector yields.

$\mathbf{x\ i + y\ j} = t^2\,\mathbf{i} + 3t\,\mathbf{j}$

(b) Proceeding as in part (a) yields

$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$

EXAMPLE

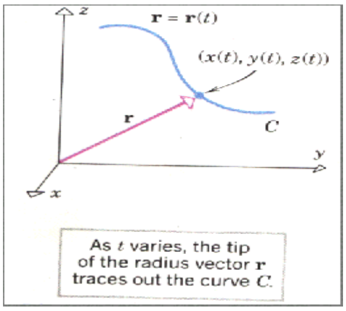
Find parametric equations that correspond to the vector equation

$\mathbf{x\ i + y\ j + z\ k = (t^3 + l)\ i + 3\ j + e^t\ k}$

Equating corresponding components yields.

$x = t^3 + 1$, $y = 3$, $z = e^t$

GRAPHS OF VECOR-VALUED FUNCTOINS

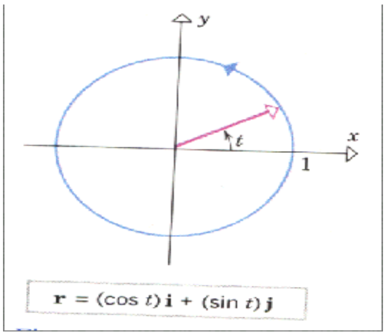


One method for interpreting a vector-valued function r(t) in 2-space or 3-space geometrically is to position the vector r = r (t) with its initial point at the origin, and let C be the curve generated by the tip of the vector r as the parameter t varies
The vector r, when positioned in this way, is called the radius vector or position vector of C, and C is called the graph of the function r (t) or, equivalently, the graph of the equation r = r (t). The vector equation r = r (t) is equivalent to a set of parametric equations, so C is also called the graph of these parametric equations.

EXAMPLE

Sketch the graph of the vector-valued function r(t) = (cos t)i + (sin t)j, $0 \le t \le 2\pi$

The graph of r (t) is the graph of the vector equation
$x\mathbf{i}+y\mathbf{j} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \le t \le 2\pi$
or equivalently, it is the graph of the parametric equations
$x = \cos t$, $y = \sin t$ $(0 \le t \le 2\pi)$
This is a circle of radius 1 that is centered at the origin with the direction of increasing t counterclockwise. The graph and a radius vector are shown in Fig.



EXAMPLE

Sketch the graph of the vector-valued function r(t) = (cos t)i+(sin t)j+2k, $0 \le t \le 2\pi$

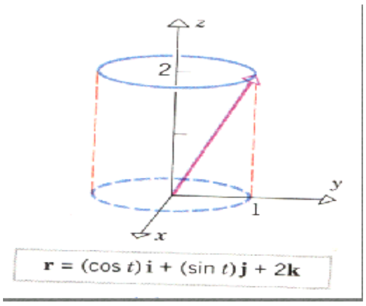
The graph of **r**(t) is the graph of the vector equation

x**i**+y**j**+z**k**=(cost)**i**+(sint)**j**+2k, $0 \le t \le 2\pi$

or, equivalently, it is the graph o the parametric equations

x=cos t, y = sin t, z=2 ($0 \le t \le 2\pi$)

From the last equation, the tip o the radius vecor traces a curve in the plane z = 2, and from the first two equations and the preceding example, the curve is a circle of radius 1 centered on the z-axis and traced counterclockwise looking down the z-axis. The graph and a radius vector are shown in Fig.



r = (cos t) i + (sin t) j + 2k

EXAMPLE

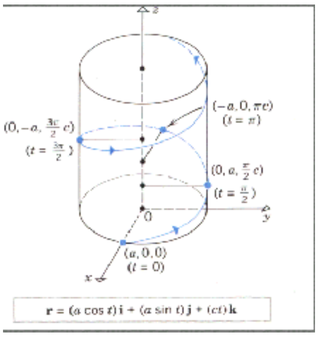
Sketch the graph of the vector-valued function r(t) = (a cos t)i + (a sin t)j + (ct)k where a and c are positive constant.

The graph of **r**(t) is the graph of the parametric equations.

x = a cos t, y = a sin t, z = ct

As the parameter t increases, the value of z = ct also increases, so the point (x, y, z) moves upward. However, as t increases, the point (x, y, z) also moves in a path directly over the circle. x = a cos t, y = a sin t in the xy-plane. The combination of these upward and circular motions produces a corkscrew-shaped curve that wraps around a right-circular cylinder of radius a centered on the z-axis.

This curve is called a circular helix.



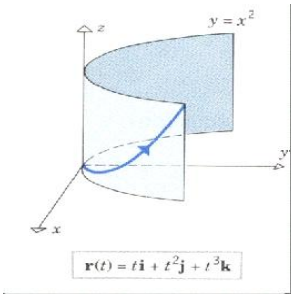
r = (a cos t) i + (a sin t) j + (ct) k

EXAMPLE

Describe the graph of the vector equation r = (– 2 + t)i + 3tj + (5 – 4t)k

The corresponding parametric equations are $x = -2 + t$, $y = 3t$, $z = 5 - 4t$
The graph is the line in 3-space that passes through the point (– 2, 0, 5) and is parallel to the vector **i** + 3**j** – 4**k**.

EXAMPLE
The graph of the vector-valued function $r(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$ is called a twisted cubic. Show that this curve lies o the parabolic cylinder $y = x^2$, and sketch the graph for $t \geq 0$
The corresponding parametric equations are $x = t$, $y = t^2$, $z = t^3$
Eliminating the parameter t in the equations for x and y yields $y = x^2$, so the curve lies on the parabolic cylinder with this equation. The curve starts at the origin for t = 0; as t increases, so do x, y, and z, so the curve is traced in the upward direction, moving away from the origin along the cylinder.



GRAPHS OF CONSTANT VECOR-VALUED FUNCTIONS
If c is a constant vector in the sense that it does not depend on a parameter, then the graph of **r** = **c** is a single point since the radius vector remains fixed with its tip at **c**.
If **c** = $x_0\mathbf{i} + y_0\mathbf{j}$ (in 2-space), then the graph is the point (x_0, y_0), and if $c = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$ (in 3-space), then the graph is the point (x_0, y_0, z_0).

EXAMPLE
The graph of the equation $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ is the point (2, 3, – 1) in 3-space.
If r(t) is a vector-valued function, then for each value of the parameter t, the expression $||r(t)||$ is a real-valued function of t because the norm (or length of r(t) is a real number.
For example,
If $\mathbf{r}(t) = t\,\mathbf{i} + (t-1)\,\mathbf{j}$
Then $||r(t)|| = \sqrt{t^2 + (t-1)^2}$ which is a real-valued function of t.

EXAMPLE
The graph of $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + 2\mathbf{k}$, $0 \leq t \leq 2\pi$
is a circle of radius 1 centered on the z-axis and lying in the plane z = 2. This circle lies on the surface of a sphere of radius $\sqrt{5}$ because for each value of t
$||r(t)|| = \sqrt{\cos^2 t + \sin^2 t + t} = \sqrt{1 + 4} = \sqrt{5}$
which shows that each point on the circle is a distance of $\sqrt{5}$ units from the origin.