Lecture No - 26 EXAMPLES

EXAMPLE

Evalaute $I=\int_0^4\int_0^{\sqrt{4y-y^2}}(x^2+y^2)dx\,dy$ by changing into polar coordinates.

The region of integration is bounded by $0\le x\le\sqrt{4y-y^2}$ and $0\le y\le 4$

Now $x=\sqrt{4y-y^2}$ is the circle $x^2+y^2-4y=0 \Rightarrow x^2+y^2=4y$.In polar coordinates this takes the form $r^2 = 4r\sin\theta$, $r = 4\sin\theta$

On changing the integral into polar coordinates, we have

$$I=\int_0^{\pi/2}\int_0^{4\sin\theta} r^2 . r\,dr\,d\theta = \int_0^{\pi/2} 64\sin^4\theta\,d\theta = 64.\frac{3.1}{4.2}.\frac{\pi}{2} = 12\pi \quad \text{(using Walli's formula)}$$

EXAMPLE

Evalaute $\iint_R e^{x^2+y^2}\,dy\,dx$.,where R is the semicircular region bounded by the x-axis and the curve $y=\sqrt{1-x^2}$

In Cartesian coordinates, the integral in question is a nonelementary integral and there is no direct way to integrate $e^{x^2+y^2}$ with respect to either x or y.

Substituting $x = r\cos\theta$, $y = r\sin\theta$, and replacing dy dx by $r\,dr\,d\theta$ enables us to evaluate the integral as

$$\iint_R e^{x^2+y^2}dy\,dx = \int_0^{\pi}\int_0^1 e^{r^2}r\,dr\,d\theta = \int_0^{\pi}\left[\frac{1}{2}e^{r^2}\right]_0^1 d\theta = \int_0^{\pi}\frac{1}{2}(e-1)d\theta = \frac{\pi}{2}(e-1).$$

EXAMPLE

Let R_a be the region bounded by the circle $x^2+y^2=a^2$. Define

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}dxdy = \lim_{a\to\infty}\iint_R e^{-(x^2+y^2)}dx\,dy$$

To evaluate this improper integral.

$$l=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\{-(x^2+y^2)\}\,dx\,dy = \lim_{a\to\infty}\iint_{Da}\exp\{-(x^2+y^2)\}\,dx\,dy$$

$$= \lim_{a\to\infty}\int_0^{2\pi}\int_0^{a}\exp\{-r^2\}r\,dr\,d\theta = \lim_{a\to\infty}\int_0^{2\pi}\frac{1}{2}(1-\exp\{-a^2\}d\theta = \lim_{a\to\infty}\frac{1}{2}(1-\exp\{-a^2\}\,\theta|_0^{2\pi}$$

$$= \pi - \lim_{a\to\infty} \frac{\pi}{\exp\{-a^2\}} = \pi$$

EXAMPLE

Prove that $\int_0^{\infty} e^{-t^2}\, dt = \frac{1}{2}\sqrt{\pi}$.

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{-(x^2+y^2)\}\, dx\, dy = \int_{-\infty}^{\infty} \exp\{-y^2\}\left[\int_{-\infty}^{\infty} \exp\{-x^2\}dx\right] dy = \left[\int_{-\infty}^{\infty} \exp\{-y^2\}dy\right]\left[\int_{-\infty}^{\infty} \exp\{-x^2\}dx\right]$$

$$= \left[\int_{-\infty}^{\infty} \exp\{-t^2\}dt\right]\left[\int_{-\infty}^{\infty} \exp\{-t^2\}\, dt\right] = \lim_{a\to\infty}\left[\int_{-a}^{a} \exp\{-t^2\}\, dt\right]^2 = 4\lim_{a\to\infty}\left[\int_0^{a} \exp\{-t^2\}\, dt\right]^2$$

Hence we have $4\lim_{a\to\infty}\left[\int_0^{a} \exp\{-t^2\}\, dt\right]^2 = \lim_{a\to\infty}\int_{-a}^{a}\int_{-a}^{a} \exp\{-(x^2+y^2)\}dx\, dy$ $4\lim_{a\to\infty}\left[\int_0^{\infty} \exp\{-t^2\}dt\right]^2 = \pi$

$$\lim_{a\to\infty}\left[\int_0^{\infty} \exp\{-t^2\}dt\right]^2 = \pi/4 \Rightarrow \int_0^{\infty} \exp\{-t^2\}\, dt = \frac{\sqrt{\pi}}{2}$$

THEOREM

Let G be the rectangular box defined by the inequalities
$a \le x \le b, \quad c \le y \le d, \qquad k \le z \le \ell$

If f is continuous on the region G, then $\iiint_G f(x, y, z)\, dV = \int_a^b \int_c^d \int_k^{\ell} f(x, y, z)\, dz\, dy\, dx$

Moreover, the iterated integral on the right can be replaced with any of the five other iterated integrals that result by altering the order of integration.

$$= \int_c^d \int_k^{\ell} \int_a^b f(x, y, z)\, dy\, dz\, dx = \int_c^d \int_a^b \int_k^{\ell} f(x, y, z)\, dy\, dx\, dz = \int_a^b \int_c^d \int_k^{\ell} f(x, y, z)\, dx\, dy\, dz$$

$$= \int_a^b \int_k^{\ell} \int_c^d f(x, y, z)\, dx\, dz\, dy = \int_k^{\ell} \int_a^b \int_c^d f(x, y, z)\, dz\, dx\, dy$$

EXAMPLE

Evalaute the triple integral $\iiint_G 12xy^2z^3\, dV$ over the rectangular box G defined by the inequalities $-1 \le x \le 2$, $0 \le y \le 3$, $0 \le z \le 2$.

We first integrate with respect to z, holding x and y fixed, then with respect to y holding x fixed, and finally with respect to x.

$$\iiint_G 12xy^2z^3\, dV = \int_{-1}^{2}\int_0^{3}\int_0^{2} 12xy^2z^3 dz dy dx = \int_{-1}^{2}\int_0^{3} \left[3xy^2z^4\right]_{z=0}^{2} dy\, dx = \int_{-1}^{2}\int_0^{3} 48xy^2\, dy\, dx$$

$$= \int_{-1}^{2} [16xy^3]_{y=0}^{3} dx = \int_{-1}^{2} 432x\, dx = 216x^2 \Big]_{-1}^{2} = 648$$

EXAMPLE

Evalaute $\iint_R\int (x - 2y + z)\, dx\, dy\, dz$ Region R : $0 \le x , 1, \; 0 \le y \le x^2, \; 0 \le z \le x + y$

$$= \int_0^1\int_0^{x^2}\int_0^{x+y} (x - 2y + z)\, dz\, dy\, dx = \int_0^1\int_0^{x^2} \left|\frac{(x - 2y + z)^2}{2}\right|_0^{x+y} dy\, dx$$

$$= \int_0^1\int_0^{x^2} \left[\frac{(x-2y+x+y)^2}{2} - \frac{(x-2y)^2}{2}\right] dy\, dx = \frac{1}{2}\int_0^1\int_0^{x^2} (3x2 - 3y2)\, dy\, dx = \frac{3}{2}\int_0^1 \left|x^2y - \frac{y^3}{3}\right|_0^{x^2} dx$$

$$= \frac{3}{2}\int_0^1 \left(x4 - \frac{x^6}{z}\right) dx = \frac{3}{2}\left|\frac{x^5}{5} - \frac{x^7}{21}\right|_0^1 = \frac{3}{2}\left[\frac{1}{5} - \frac{1}{21}\right] = \frac{8}{35}$$

Example:

Evalaute $\iiint_S xyzd\, x\, dy\, dz$ Where S = $\{(x,y,z): x^2+y^2+z^2 \le 1, x \ge 0, y \ge 0, z \ge 0\}$

S is the sphere $x^2 + y^2 + z^2 = 1$ .Since x, y, z are al +ve so we have to consider only the +ve octant of the sphere.

Now $x^2 + y^2 + z^2 = 1$. So that $z = \sqrt{1 - x^2 - y^2}$

The Projection of the sphere on xy plan is the circle $x^2 + y^2 = 1$.

This circle is covered as y-varies from 0 to $\sqrt{1 - x^2}$ and x varies from 0 to 1.

$$\iiint_R xyz\, dx\, dy\, dz = \int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyz\, dz\, dy\, dx = \int_0^1\int_0^{\sqrt{1-x^2}} xy \left|\frac{z^5}{2}\right|_0^{\sqrt{1-x^2-y^2}} dy\, dx$$

$$= \int_0^1\int_0^{\sqrt{1-x^2}} xy\left(\frac{1 - x^2 - y^2}{2}\right) dy\, dx = \frac{1}{2}\int_0^1\int_0^{\sqrt{1-x^2}} x\,(y - x^2y - y^3)\, dy\, dx$$

$$= \frac{1}{2}\int_0^1 x\left(\frac{y^2}{2} - \frac{x^2y^2}{2} - \frac{y^4}{4}\right)\Big|_0^{\sqrt{1-x^2}} dx = \frac{1}{4}\int_0^1 x\left[1-x^2-x^2\,(1-x^2)-\frac{(1-x^2)^2}{2}\right] dx$$

$$= \frac{1}{8}\int_0^1 (x - 2x^3 + x^5)\, dx = \frac{1}{8}\left|\frac{x^2}{2} - \frac{x^4}{2} + \frac{x^6}{6}\right|_0^1 = \frac{1}{8}\left(\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right) = \frac{1}{48}$$