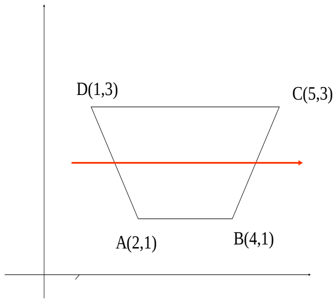
Lecture No -22 Examples

EXAMPLE

$$\iint_R xy\, dA$$

R is the region bounded by the Trapezium with the vertices (1, 3) (5, 3) (2, 1) (4, 1)



Slope of AD = $\dfrac{3-1}{1-2} = -2$

Equation of line AD

$y - 1 = -2(x - 2)$

$y - 1 = -2x + 4$

$-2x = y - 5$

$x = -\dfrac{y-5}{2}$

Slope of BC = $\dfrac{3-1}{5-4} = 2$

Equation of line BC

$y - 1 = 2(x - 4) \Rightarrow y - 1 = 2x - 8 \Rightarrow 2x = y + 7 \Rightarrow x = \dfrac{y+7}{2}$

$$\int_1^3 \int_{-(y-5)/2}^{(y+7)/2} xy\, dx\, dy = \int_1^3 y \left| x^2 \right|_{-(y-5)/2}^{(y+7)/2} y\, dy \qquad = \int_1^3 (3y^2 + 3y)\, dy$$

$$= \left| y^3 + \frac{3y^2}{2} \right|_1^3 = (3)^3 + \frac{3(3)^2}{2} - 1 - \frac{3}{2} = 38$$

EXAMPLE

Use double integral to find the volume of the wedge cut from the cylinder $4x^2+y^2=9$ by the plane z=0 and z=y+3

Solution:

Since we can write $4x^2 + y^2 = 9$ as $\dfrac{x^2}{\left(\frac{3}{2}\right)^2} + \dfrac{y^2}{9} = 1$ this is equation of ellipse.

Now the Lower and upper limits for "x" are $x = \dfrac{-\sqrt{9-y^2}}{2}$ *and* $x = \dfrac{\sqrt{9-y^2}}{2}$

And upper and lower limits for "x" are -3 and 3 respectively. So the required volume is given by $\displaystyle\int_{-3}^{3} \int_{-\frac{\sqrt{9-y^2}}{2}}^{\frac{\sqrt{9-y^2}}{2}} (y+3)\, dx\, dy$

$$= \int_{-3}^{3} \left| xy + 3x \right|_{-\frac{\sqrt{9-y^2}}{2}}^{\frac{\sqrt{9-y^2}}{2}} dy$$

$$= \int_{-3}^{3} \left[y\left(\frac{\sqrt{9-y^2}}{2}\right) + 3\left(\frac{\sqrt{9-y^2}}{2}\right) + y\left(\frac{\sqrt{9-y^2}}{2}\right) + 3\left(\frac{\sqrt{9-y^2}}{2}\right) \right] dy$$

$$= \int_{-3}^{3} \left[y\sqrt{9-y^2} + 3\sqrt{9-y^2} \right] dy$$

$$= -\frac{1}{2}\int_{-3}^{3} \sqrt{9-y^2}\,(-2y)dy + 3\int_{-3}^{3} \sqrt{9-y^2}\,dy$$

$$= -\frac{1}{2}.\frac{2}{3}\left|(9-y^2)^{\frac{3}{2}}\right|_{-3}^{3} + 3\int_{-3}^{3} \sqrt{9-y^2}\,dy$$

$$= -\frac{1}{3}.[0] + 3\int_{-3}^{3} \sqrt{9-y^2}\,dy = 3\left|\frac{y}{2}\sqrt{9-y^2} + \frac{9}{2}\sin^{-1}(\frac{y}{2})\right|_{-3}^{3} = \frac{27\pi}{2}$$

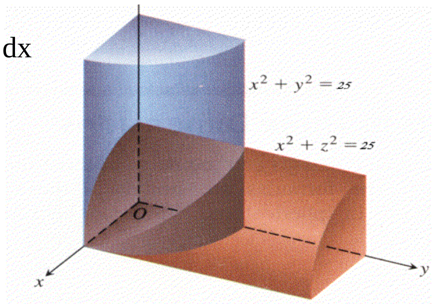
EXAMPLE

Use double integral to find the volume of solid common to the cylinders $x^2+ y^2=25$ and $x^2+ z^2=25$

$$\text{Volume} = 8\int_0^5 \int_0^{\sqrt{25-x^2}} \sqrt{25-x^2}\; dy\, dx$$

$$= 8\int_0^5 \sqrt{25-x^2}\; \left| y \right|_0^{\sqrt{25-x^2}} dx$$

$$= 8\int_0^5 (25-x^2)\, dx$$



$$= 8\left|25x - \frac{x^3}{3}\right|_0^5 = 8\left(125 - \frac{125}{3}\right) = 8\left(\frac{375-125}{3}\right) = 8\left(\frac{250}{3}\right) = \frac{2000}{3}$$

AREA CALCUALTED AS A DOUBLE INTEGRAL

$$V = \iint_R 1\, dA = \iint_R dA \qquad (1)$$

However, the solid has congruent cross sections taken parallel to the xy-plan so that

V = area of base x height=area of R.1 = area of R

Combining this with (1) yields the area formula

$$\text{area of R} = \iint_R dA \qquad (2)$$

EXAMPLE

Use a double integral to find the area of the region R enclosed between the parabola

$y = ½\, x^2$ and the line y=2x

$$\text{area of R} = \iint_R dA$$

$$= \int_0^4 \int_{x^2/2}^{2x} dy\, dx$$

$$= \int_0^4 [y]_{y=x^2/2}^{2x} dx = \int_0^4 \left(2x - \frac{1}{2}x^2\right) dx = \left[x^2 - \frac{x^3}{6}\right]_0^4 = \frac{16}{3}$$

Treating R as type IIyields.

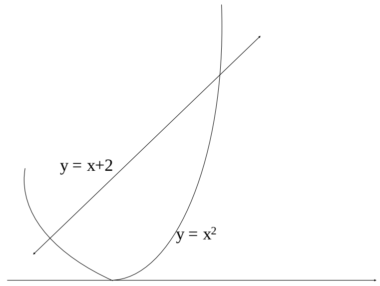
$$\text{area of } R = \iint_R dA = \int_0^8 \int_{y/2}^{\sqrt{2y}} dx\, dy$$

$$= \int_0^8 [x]_{y=x^2/2}^{\sqrt{2x}} dx = \int_0^8 \left(\sqrt{2y} - \frac{1}{2}y\right) dy = \left[\frac{2\sqrt{2}}{3} y^{3/2} - \frac{y^2}{4}\right]_0^8 = \frac{16}{3}$$

EXAMPLE

Find the area of the region R enclosed by the parabola $y=x^2$ and the line $y=x+2$

$$\int_{-1}^{2} \int_{y=x^2}^{y=x+2} dy\, dx$$



$$= \int_{-1}^{2} [y]_{x^2}^{x+2} dx$$

$$= \int_{-1}^{2} [x + 2 - x^2] dx$$

$$= \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2} = \frac{9}{2}$$