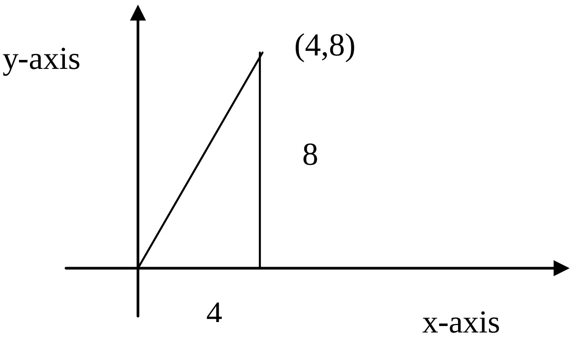
Lecture No -19 Use Of Integrals

Area as anti-derivatives

$$\int_0^4 2x\,dx = \left|x^2\right|_0^4$$
$$= (4)^2 = 16$$



Area of triangle =1/2 base x altitude
= ½ (4) (8) = 16

volume as anti-derivative

$$\text{Volume} = \int_0^2\int_0^3 5\,dydx$$
$$= \int_0^2 \left|5y\right|_0^3 dx = \int_0^2 15\,dx$$
$$= \left|15x\right|_0^2 = 30$$

$0 \le x \le 2,\quad 0 \le y \le 3,\quad 0 \le z \le 5$
Volume = 2 x 3 x 5 = 30
The following results are analogous to the result of the definite integrals of a function of single variable.

$$\iint_R cf(x,y)\,dxdy$$
$$= c\iint_R f(x,y)\,dxdy \quad (\text{ c a constant })$$
$$\iint_R [f(x,y) + g(x,y)]\,dxdy$$
$$= \iint_R f(x,y)\,dxdy + \iint_R g(x,y)\,dxdy$$
$$\iint_R [f(x,y) - g(x,y)]\,dxdy$$
$$= \iint_R f(x,y)\,dxdy - \iint_R g(x,y)\,dxdy$$

Use double integral to find the volume under the surface $z = 3x^3 + 3x^2y$ and the rectangle $\{(x,y):1\le x \le 3,\ 0\le y \le 2\}$.

$$\text{Volume} = \int_0^2\int_1^3 (3x^3 + 3x^2y)\,dx\,dy$$
$$= \int_0^2 \left|\frac{3x^4}{4} + x^3y\right|_1^3 dy$$
$$= \int_0^2 \left[\frac{3(3)^4}{4} - \frac{3}{4} + (3)^3y - y\right] dy$$
$$= \int_0^2 [60 + 26\,y]\,dy$$
$$= \left|60y + 13\,y^2\right|_0^2 = 172$$

Use double integral to find the volume of solid in the first octant enclosed by the surface z = x2 and the planes x=2, y =0, y = 3 and z = 0

$$\text{Volume} = \int_0^2 \int_0^3 x^2 \, dy dx$$

$$= \int_0^2 \left| x^2 y \right|_0^3 dx = \int_0^2 [3x^2] \, dx$$

$$= \left| x^3 \right|_0^2 = 8$$

$$\iint_R f(x,y)dA \geq 0 \text{ if } f(x,y) \geq 0 \text{ on } R$$

$$\iint_R f(x,y)dA \geq \iint_R cg(x,y)dA$$

$$\text{if } f(x,y) \geq g(x,y)$$

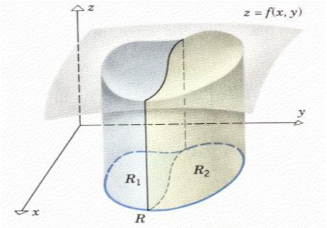
If $f(x,y)$ is nonnegative on a region R, then subdividing R into two regions R_1 and R_2 has the effect to subdividing the solid between R and z= $f(x,y)$ into two solids, the sum of whose volumes is the volume of the entire solid

$$\iint_R f(x, y) \, dA = \iint_{R1} f(x, y) \, dA + \iint_{R2} f(x, y) dA$$

The volume of the solid S can also be obtained using cross sections perpendicular to the y -axis.

$$\text{Vol } (S) = \int_c^d A(y) \, dy \qquad (1)$$

Where A(y) represents the area of the cross section perpendicular to the y-axis taken at the point y

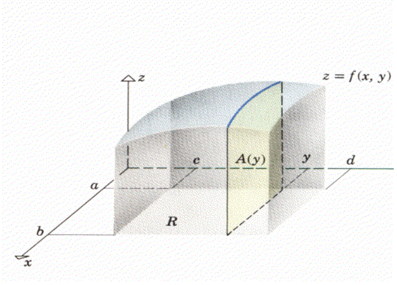


The volume of the entire solid is the sum of the volumes of the solids above R_1 and R_2.

<u>How to compute cross sectional area</u>

For each fixed y in the interval $c \leq y \leq d$, the function f(x,y) is a function of x alone ,and A(y)may be viewed as the area under the graph of this function along the interval a<x<b, Thus

$$A(y) = \int_a^b f(x, y) \, dx$$



Substiuting this expression in (1) yields.

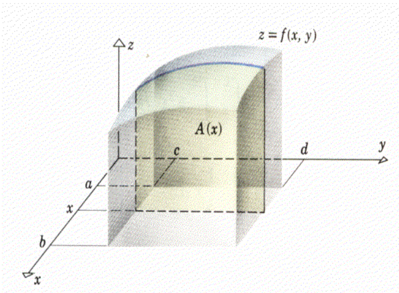
$$\text{Vol } (S) = \int_c^d \left[\int_a^b f(x, y) \, dx \right] dy$$

$$= \int_c^d \int_a^b f(x, y) \, dx \, dy$$

Similarly the volume of the side S can also be obtained using sections perpendicular to the x-axis

$$\text{Vol } (S) = \int_a^b A(x) \qquad (3)$$

Where A(x) is the area of the cross section perpendicular to the x-axis taken at the point x.



For each fixed x in the interval $a \le x \le b$ the function f(x,y) is a function of alone, so that the area A(x) is given by

$$A(x) = \int^{d} f(x,y)dy$$

Substituting this expression in
(3) yields

$$\text{Vol}(S) = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx$$

$$= \int_a^b \int_c^d f(x,y)\,dy\,dx \qquad (4)$$

From eq (2) and eq (4) we have

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

$$= \int_a^b \int_c^d f(x,y)\,dy\,dx$$

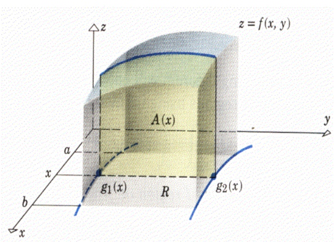
<u>Double integral for non-rectangular region</u>

Type I region is bounded the left and right by the vertical lines x=a and x=b and is bounded below and above by continuous curves $y=g_{1(x)}$ and $y=g_{2(x)}$ where

$$g_{1(x)} \le g_{2(x)} \text{ for } a \le x \le b$$

If R is a type I region on which f(x, y) is continuous, then

$$\iint_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_{2(x)}} f(x,y)dydx \quad (1)$$



By the method of cross section, the volume of S is also given by

$$\text{Vol (S)} = \int_a^b A(x)\,dx \qquad (2)$$

where A(x) is the area of the cross section at the fixed point x this cross section area extends from $g_{1(x)}$ to $g_{2(x)}$ in the y-direction,

$$\text{so} \quad , A(x) = \int_{g_1(x)}^{g_2(x)} f(x, y)\,dy$$

Substituting this in (2) we obtain

$$\text{Vol (S)} = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\,dy\,dx$$

Since the volume of S is also given by

$$\iint_R f(x, y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\,dy\,dx$$

Type II region is bounded below and above by horizontal lines y=c and y=d and is bounded in the left and right by continuous curves $x=h_1(y)$ and $x=h_2(y)$ satisfying $h_1(y) \le h_2(y)$ for for $c \le y \le d$.

. If R is a type II region on which f (x, y) is continuous, then

$$\iint_R f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dx\,dy$$

Similarly, the partial definite integral with respect to $\int_c^d f(x,y)$ is evaluated by holding x fixed and integrating with respect to y.

An integral of the form $\int_c^d f(x, y)\,dy$ produces a function of x.