Lecture No -18 Revision of Integration

Example:

Consider the following integral $\int_0^1 (xy+y^2)dx$ Integrating we get

$$\int_0^1 (xy+y^2)dy = x\int_0^1 ydx + \int_0^1 y^2 dx$$

$$= x\left|\frac{y^2}{2}\right|_0^1 + \left|\frac{y^3}{3}\right|_0^1 = y(\frac{1}{2}) + y^2$$

$$\Rightarrow \int_0^1 (xy+y^2)dx = y(\frac{1}{2}) + y^2$$

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$$\Rightarrow \int_0^1 (xy+y^2)dx = \frac{x}{2} + \frac{1}{3}$$

Double Integral

Symbolically, the double integral of two variables “x” and “y” over the certain region R of the plane is denoted by $\iint_R f(x,y)dxdy$.

Example:

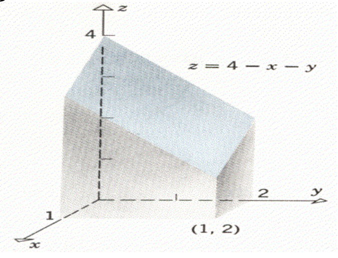
Use a double integral to find out the solid bounded above by the plane Z = 4 – x – y and below by the rectangle $R = \{(x,y): 0 \le x \le 1, 0 \le y \le 2\}$

Solution:

We have to find the region “R”out the volume “V” over that is,

$$V = \iint_R (4 - x - y)dA$$

And the solid is shown in the figure below.



$$V=\iint_R (4-x-y)dA=\int_0^2\int_0^1 (4-x-y)dxdy$$

$\int_0^2\int_0^1 (4-x-y)dxdy=\int_0^2 \left|4x-\frac{x^2}{2}-xy\right|_{x=0}^{x=1} dy$ After putting the upper and lower limits we get

$\int_0^2\left(\frac{7}{2}-y\right)dy=\left|\frac{7}{2}y-\frac{y^2}{2}\right|_0^2$ again after putting the limits we get the required volume of the solid $V=\iint_R (4-x-y)dA=5$.

Example:

Evaluate the double integral $\int_0^1\int_0^1 (xy+y^2)dxdy$

Solution:

First we will integrate the given function with respect to "x' and our integral becomes $\int_0^1\int_0^1 (xy+y^2)dydx=\int_0^1\left(x\left|\frac{y^2}{2}\right|_0^1+\left|\frac{y^3}{3}\right|_0^1\right)dy$

and after applying the limits we have,

$$\int_0^1\int_0^1 (xy+y^2)dydx=\int_0^1\left(\frac{x}{2}+\frac{1}{3}\right)dy \quad integrating\ we\ get$$

$$\int_0^1\int_0^1 (xy+y^2)dydx=\left|\frac{x^2}{4}+\frac{x}{3}\right|_0^1=\frac{1}{4}+\frac{1}{3}=\frac{7}{12}$$

Iterated or Repeated Integral

The expression $\int_c^d\left[\int_a^b f(x,y)dx\right]dy$ is called iterated or repeated integral. Often the brackets are omitted and this expression is written as

$\int_c^d\int_a^b f(x,y)dxdy=\int_c^d\left[\int_a^b f(x,y)dx\right]dy$ Where we have $\int_a^b f(x,y)dx$ yields a function of "y", which is then integrated over the interval $c\le y\le d$.

Similarly $\int_a^b\int_c^d f(x,y)dydx=\int_a^b\left[\int_c^d f(x,y)dy\right]dx$ Where we have $\int_c^d f(x,y)dy$ yields a function of "x"which is then integrated the interval $a\le x\le b$.

Example:

Evaluate the integral $\int_0^1\int_0^2 (x+3)dydx$.

Solution:

Here we will first integrate with respect to "y" and get a function of "x" then we will integrate that function with respect to "x" to get the required answer. So

$\int_0^1\int_0^2 (x+3)dydx = \int_0^1 (x+3)\left|y\right|_0^2 dx$ and after putting the limits we get, $\int_0^1 (x+3)\left|y\right|_0^2 dx = \int_0^1 2(x+3)dx = 2\left|\frac{x^2}{2}+3x\right|_0^1$ and the required value of the double integral is $\int_0^1\int_0^2 (x+3)dydx = 2(\frac{1}{2}+3) = 7$.

Now if we change the order of integration so we get $\int_0^2\int_0^1 (x+3)dxdy$ Then we have $\int_0^2\int_0^1 (x+3)dxdy = \int_0^2 \left|(\frac{x^2}{2}+3)\right|_0^1 dy = \int_0^2 \frac{7}{2}dy = \frac{7}{2}\left|y\right|_0^2 = 7$. Now you note that the value of the integral remain same if we change the order of integration. Actually we have a stronger result which we sate as a theorem.

Theorem:

Let R be the rectangle defined by the inequalities **a < x < b** and **c < y < d.** If f(x, y) is continuous on this rectangle, then $\iint_R f(x,y)dA = \int_c^d\int_a^b f(x,y)dxdy = \int_a^b\int_c^d f(x,y)dxdy$.

Remark:

This powerful theorem enables us to evaluate a double integral over a rectangle by calculating an iterated integral. Moreover the theorem tells us that the **"order of integration in the iterated integral does not matter".**

Example:

Evaluate the integral $\int_0^{\ln 2}\int_0^{\ln 3} e^{x+y}dxdy$

Solution:

First we will integrate the function with respect to "x". Note that we can write e^{x+y} as $e^x.e^y$ So we have, $\int_0^{\ln 2} e^y\left|e^x\right|_0^{\ln 3} dy = \int_0^{\ln 2} e^y(3-1)dy$ Here we use the fact that "e" and "ln" are inverse function of each other. So we have $e^{\ln 3} = 3$. Thus we get,

$\int_0^{\ln 2} e^y\left|e^x\right|_0^{\ln 3} dy = 2\int_0^{\ln 2} e^y dy = 2\left|e^y\right|_0^{\ln 2} = 2(2-1) = 2 \; is\ the\ \ required\ \ answer.$

Example:

Evaluate the integral $\int_0^{\ln 3}\int_0^{\ln 2} e^{x+y}dydx$ **(Note that in this example we change the order of integration)**

Solution:

First we will integrate the function with respect to "y". Note that we can write e^{x+y} as $e^x.e^y$ So we have, $\int_0^{\ln 3} e^x\left|e^y\right|_0^{\ln 2} dy = \int_0^{\ln 3} e^x(2-1)dy$ Here we use the fact that "e" and "ln" are inverse function of each other. So we have $e^{\ln 2} = 2$. Thus we get,

$$\int_0^{\ln 3} e^x \left|e^y\right|_0^{\ln 2} dx = \int_0^{\ln 3} e^x dy = \left|e^x\right|_0^{\ln 3} = (3-1) = 2 \ \textit{is the required answer}.$$

Note that in both cases our integral has the same value.

Over view:

Double integrals **Page # 854-857**

Exercise Set 17.1 (page 857): 1,3,5,7,9,11,13,15,17,19