Lecture No -17 Examples

EXAMPLE

Find the absolute maximum and minimum values of f(x,y)=xy-x-3y on the closed triangular region R with vertices (0, 0), (0, 4), and (5, 0).

$$f(x,y) = xy - x - 3y \qquad (1)$$

$f_x(x, y) = y - 1, \quad f_y(x, y) = x - 3$

For critical points

$f_x(x, y) = 0, \quad y - 1 = 0$

$$y = 1 \qquad (2)$$

$f_y(x, y) = 0, \quad 3x - 3 = 0$

$$x = 3 \qquad (3)$$

Thus, (3, 1) is the only critical point in the interior of R. Next, we want to determine the location of the points on the boundary of R at which the absolute extrema might occur. The boundary of R consists of three line segments, each of which we shall treat separately.

(i) The line segment between (0, 0) and (5,0)

On this line segment we have y = 0, so (1) simplifies to a function of the single variable x,

$$u(x) = f(x, 0) = -x, \; 0 \le x \le 5 \qquad (4)$$

The function has no critical points because the $u'(x)=-1$ is non zero for all x.Thus,the exteme values of u(x) occure at the endpoints x=0 and x=5 , which corresponds to the points (0,0) and (5,0) of R.

ii) The line segment between (0,0) and (0,4)

On this line segment we have x = 0, so (1) simplifies to a function of the single variable y,

$$v(y) = f(0, y) = -3y, \; 0 \le y \le 4. \qquad (5)$$

This function has no critical points because $v'(y)= -3$ is nonzero for all y. Thus, the extreme values of v(y) occur at the endpoints y =0 and y=4 ,which correspond to the point (0,0) and (0,4) or R.

iii) The line segment between (5,0) and (0,4)

In the xy-plan, an equation is

$$y = -\frac{4}{5}x + 4, \; 0 \le x \le 5 \qquad (6)$$

so (1) simplifies to a function of the single variable x,

$$w(x) = f\left(x, -\frac{4}{5}x + 4\right)$$

$$= x\left(-\frac{4}{5}x + 4\right) - x - 3\left(-\frac{4}{5}x + 4\right)$$

$$= -\frac{4}{5}x^2 + \frac{27}{5}x - 12$$

$$w'(x) = -\frac{8}{5}x + \frac{27}{5}$$

$$by\ w'(x) = 0,\ \ we\ get\ x = \frac{27}{8}$$

This shows that $x = \frac{27}{8}$ is the only critical point of w. Thus, the extreme values of w occur either at the critical point $x = \frac{27}{8}$ or at the endpoints x = 0 and x = 5. The endpoints correspond to the points (0, 4) and (5, 0) of R, and from (6) the critical point corresponds to $\left[\frac{27}{8}, \frac{13}{10}\right]$

(x, y)	(0, 0)	(5, 0)	(0, 4)	(27/8, 13/10)	(3, 1)
$f(x, y)$	0	-5	-12	-231/80	-3

Finally, from the table below, we conclude that the absolute maximum value of f is f (0,0) = 0 and the absolute minimum value is $f(0,4)$=-12

Example

Find three positive numbers whose sum is 48 and such that their product is as large as possible

Let x,y and z be the required numbers, then we have to maximize the product

$$\mathbf{f(x,y)=xy(48-x-y)}$$

Since

$$f_x=48y-2xy-y^2 \quad , \quad f_y=48x-2xy-x^2$$

solving

$$f_x=0 \quad , \quad f_y=0$$

we get $x=16,\ \ y=16,\ \ z=16$

Since $x+y+z=48$

$f_{xx}(x,y) = -2y$, $\quad f_{xx}(16, 16) = -32 < 0$
$f_{xy}(x, y) = 48-2x-2y$, $\quad f_{xy}(16, 16) = -16$
$f_{yy}(x, y) = -2x$, $\quad f_{yy}(16, 16) = -32$
$D=f_{xx}(16,16)f_{yy}(16,16)-f^2_{xy}(16,16)$

$$= (-32)(-32) - (16)^2 = 768 > 0$$

For x = 16, y = 16 we have z = 16 since x + y + z = 48

Thus, the required numbers are 16, 16, 16.

Example

Find three positive numbers whose sum is 27 and such that the sum of their squares is as small as possible

Let x, y, z be the required numbers, then we have to

f(x,y) = x² + y² + z²

$= x^2 + y^2 + (27 - x - y)^2$

Since x+y+z = 27

$f_x = 4x+2y-54, \quad f_y = 2x+4y-54,$

$f_{xx} = 4, \quad f_{yy} = 4, \quad f_{xy} = 2$

Solving $f_x = 0, \quad f_y = 0$

We get x = 9, y = 9, z = 9

Since x + y + z = 27

$\mathbf{D = f_{xx}(9, 9)\, f_{yy}(9, 9) - [f_{xy}(9, 9)]^2}$

$= (4)(4) - 2^2 = 12 > 0$

This shows that f is minimum

x = 9, y = 9, z = 9, so the required numbers are 9, 9, 9.

Example

Find the dimensions of the rectangular box of maximum volume that can be inscribed in a sphere of radius 4.

Solution:

The volume of the parallelepiped with dimensions x, y, z is

V = xyz

Since the box is inscribed in the sphere of radius 4, so equation of sphere is $x^2 + y^2 + z^2 = 4^2$ from this equation we can write $z = \sqrt{16 - x^2 - y^2}$ and putting this value of "z" in above equation we get $V = xy\sqrt{16 - x^2 - y^2}$ .Now we want to find out the maximum value of this volume, for this we will calculate the extreme values of the function "V". For extreme values we will find out the critical points and for critical points we will solve the equations $V_x = 0$ and $V_y = 0$ .Now we have

$$V_x = y\sqrt{16 - x^2 - y^2} + \frac{xy(-2x)}{2\sqrt{16 - x^2 - y^2}}$$

$$\Rightarrow V_x = y\left\{\frac{-2x^2 - y^2 + 16}{\sqrt{16 - x^2 - y^2}}\right\} \; Now \;\; V_x = 0 \Rightarrow y\left\{\frac{-2x^2 - y^2 + 16}{\sqrt{16 - x^2 - y^2}}\right\} = 0$$

$$\Rightarrow -2x^2 - y^2 + 16 = 0 \Rightarrow 2x^2 + y^2 = 16 \ldots\ldots\ldots\ldots\ldots\ldots(a)$$

Similarly we have

$$V_y = x\sqrt{16 - x^2 - y^2} + \frac{xy(-2y)}{2\sqrt{16 - x^2 - y^2}}$$

$$\Rightarrow V_y = x\left\{\frac{-x^2 - 2y^2 + 16}{\sqrt{16 - x^2 - y^2}}\right\} \; Now \;\; V_y = 0 \Rightarrow x\left\{\frac{-x^2 - 2y^2 + 16}{\sqrt{16 - x^2 - y^2}}\right\} = 0$$

$$\Rightarrow -x^2 - 2y^2 + 16 = 0 \Rightarrow x^2 + 2y^2 = 16 \ldots\ldots\ldots\ldots\ldots\ldots(b)$$

Solving equations (a) and (b) we get the $x=\frac{4}{\sqrt{3}}$ *and* $y=\frac{4}{\sqrt{3}}$

Now $V_{xx}=\frac{xy(2x^2+3y^2-48)}{(16-x^2-y^2)^{\frac{3}{2}}}$ (We obtain this by using quotient rule of differentiation)

$V_{xx}(\frac{4}{\sqrt{3}},\frac{4}{\sqrt{3}})=-\frac{16}{\sqrt{3}}\prec 0$

Also we have to calculate $V_{yy}=\frac{xy(3x^2+2y^2-48)}{(16-x^2-y^2)^{\frac{3}{2}}}$ and $V_{yy}(\frac{4}{\sqrt{3}},\frac{4}{\sqrt{3}})=-\frac{16}{\sqrt{3}}\prec 0$ Also note that $V_{xy}(\frac{4}{\sqrt{3}},\frac{4}{\sqrt{3}})=-\frac{8}{\sqrt{3}}$ Now as we have the formula for the second order partial derivative is $f_{xx}.f_{yy}-(f_{xy})^2$ and putting the values which we calculated above we note that $f_{xx}(\frac{4}{\sqrt{3}},\frac{4}{\sqrt{3}}).f_{yy}(\frac{4}{\sqrt{3}},\frac{4}{\sqrt{3}})-(f_{xy}(\frac{4}{\sqrt{3}},\frac{4}{\sqrt{3}}))^2=+\frac{320}{3}\succ 0$ Which shows that the function V has maximum value when $x=\frac{4}{\sqrt{3}}$ *and* $y=\frac{4}{\sqrt{3}}$. So the dimension of the rectangular box are $x=\frac{4}{\sqrt{3}}, y=\frac{4}{\sqrt{3}}$ *and* $z=\frac{4}{\sqrt{3}}$.

Example

A closed rectangular box with volume of 16 ft^3 is made from two kinds of materials. The top and bottom are made of material costing Rs. 10 per square foot and the sides from material costing Rs.5 per square foot. Find the dimensions of the box so that the cost of materials is minimized

Let x, y, z, and C be the length, width, height, and cost of the box respectively. Then it is clear form that

C=10(xy+xy)+5(xz+xz)+5(yz+yz)---------------(1)

C=20xy+10(x+y)z

The volume of the box is given by

xyz=16----------------------------------(2)

Putting the value of z from (2) in (1), we have

$C = 20xy + 10(x + y)\frac{16}{xy}$

$C = 20xy + \frac{160}{y} + \frac{160}{x}$

$C_x=20y - \frac{160}{x^2}$, $C_y = 20x - \frac{160}{y^2}$

For critical points

$C_x = 0$

$20y - \frac{160}{x^2} = 0$ and $C_y = 0$

$20x - \frac{160}{y^2} = 0$

Solving these equations, we have $x = 2$, $y = 2$. Thus the critical point is (2, 2).

$C_{xx}(x, y) = \frac{320}{x^3}$

$C_{xx}(2, 2) = \frac{320}{8} = 40 > 0$

$C_{yy}(x, y) = \frac{320}{y^3}$

$C_{yy}(2, 2) = \frac{320}{8} = 40$

$C_{xy}(x, y) = 20$

$C_{xy}(2, 2) = 20$

$C_{xx}(2,2)\, C_{yy}(2,2) - C^2_{xy}(2,2) = (40)(40)-(20)^2 = 1200 > 0$

This shows that S has relative minimum at x = 2 and y = 2. Putting these values in (2), we have z = 4, so when its dimensions are $2 \times 2 \times 4$.

Example

Find the dimensions of the rectangular box of maximum volume that can be inscribed in a sphere of radius a.

Solution:

The volume of the parallelepiped with dimensions x, y, z is

$$V = xyz$$

Since the box is inscribed in the sphere of radius 4, so equation of sphere is

$x^2 + y^2 + z^2 = 4^2$ from this equation we can write $z = \sqrt{a^2 - x^2 - y^2}$ and putting this value of "z" in above equation we get $V = xy\sqrt{a^2 - x^2 - y^2}$ .Now we want to find out the maximum value of this volume, for this we will calculate the extreme values of the function "V". For extreme values we will find out the critical points and for critical points we will solve the equations $V_x = 0$ and $V_y = 0$ .Now we have

$$V_x = y\sqrt{a^2 - x^2 - y^2} + \frac{xy(-2x)}{2\sqrt{a^2 - x^2 - y^2}}$$

$$\Rightarrow V_x = y\left\{\frac{-2x^2 - y^2 + a^2}{\sqrt{a^2 - x^2 - y^2}}\right\} \; Now \; V_x = 0 \Rightarrow y\left\{\frac{-2x^2 - y^2 + a^2}{\sqrt{a^2 - x^2 - y^2}}\right\} = 0$$

$$\Rightarrow -2x^2 - y^2 + 16 = 0 \Rightarrow 2x^2 + y^2 = a^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(a)$$

Similarly we have

$$V_y = x\sqrt{a^2-x^2-y^2} + \frac{xy(-2y)}{2\sqrt{a^2-x^2-y^2}}$$

$$\Rightarrow V_y = x\left\{\frac{-x^2-2y^2+a^2}{\sqrt{a^2-x^2-y^2}}\right\} \ Now\ V_y = 0 \Rightarrow x\left\{\frac{-x^2-2y^2+a^2}{\sqrt{a^2-x^2-y^2}}\right\} = 0$$

$$\Rightarrow -x^2-2y^2+a^2 = 0 \Rightarrow x^2+2y^2 = a^2 ....(b)$$

Solving equations (a) and (b) we get the $x = \frac{a}{\sqrt{3}}\ and\ y = \frac{a}{\sqrt{3}}$

Now $V_{xx} = \frac{xy(2x^2+3y^2-3a^2)}{(a^2-x^2-y^2)^{\frac{3}{2}}}$ (We obtain this by using quotient rule of differentiation)

$$V_{xx}\left(\frac{a}{\sqrt{3}},\frac{a}{\sqrt{3}}\right) = -\frac{a^2}{\sqrt{3}} \prec 0$$

Also we have to calculate $V_{yy} = \frac{xy(3x^2+2y^2-3a^2)}{(a^2-x^2-y^2)^{\frac{3}{2}}}$ and $V_{yy}\left(\frac{a}{\sqrt{3}},\frac{a}{\sqrt{3}}\right) = -\frac{a^2}{\sqrt{3}}$ Also note that $V_{xy}\left(\frac{a}{\sqrt{3}},\frac{a}{\sqrt{3}}\right) = -\frac{2a}{\sqrt{3}}$ Now as we have the formula for the second order partial derivative is $f_{xx}.f_{yy} - (f_{xy})^2$ and putting the values which we calculated above we note that $f_{xx}\left(\frac{a}{\sqrt{3}},\frac{a}{\sqrt{3}}\right).f_{yy}\left(\frac{a}{\sqrt{3}},\frac{a}{\sqrt{3}}\right) - \left(f_{xy}\left(\frac{a}{\sqrt{3}},\frac{a}{\sqrt{3}}\right)\right)^2 = +\frac{20a^2}{3} \succ 0$ Which shows that the function V has maximum value when $x = \frac{a}{\sqrt{3}}\ and\ y = \frac{a}{\sqrt{3}}$. So the dimension of the rectangular box are $x = \frac{a}{\sqrt{3}}, y = \frac{a}{\sqrt{3}}\ and\ z = \frac{a}{\sqrt{3}}$.

Example:

Find the points o the plane x + y + z = 5 in the first octant at which f(x,y,z) = xy^2z^2 has maximum value.

Solution:

Since we have f(x,y,z) = xy^2z^2 and we are given the plane x + y + z = 5 from this equation we can write x = 5 – y – z . Thus our function "f" becomes
f((5 – y – z),y,z) = (5 – y – z)y^2z^2 Say this function u(y,z) That is u(y,z) = (5 – y – z)y^2z^2
Now we have to find out extrema of this function. On simplification we get
u(y,z) = 5 $y^2z^2 - y^3z^2 - y^2z^3$

$u_y = 10yz^2 - 3y^2z^2 - 2yz^3$
$\quad = yz^2(10 - 3y - 2z)$
$u_z = 10y^2z - 2y^3z - 3y^2z^2$
$\quad = y^2z\,(10 - 2y - 3z)$
$u_y = 0, \quad u_z = 0$
$y = 0, \quad z = 0$
$10 - 3y - 2z = 0$
$10 - 2y - 3z = 0$

On solving above equations we get $-10 + 5z = 0 \Rightarrow z = 2$ and $10 - 3y - 4 = 0 \Rightarrow y = 2$

$u_{yy} = 10z^2 - 6yz^2 - 2z^3$
$u_{zz} = 10y^2 - 2y^3 - 6y^2z$
$u_{yz} = 20yz - 6y^2z - 6yz^2$
at
$y = 2, \quad z = 2$
$u_{yy}(2,2) = 40 - 48 - 16 = -24 < 0$
$y_{zz}(2,2) = 40 - 16 - 48 = -24$
$u_{yz}(2,2) = 80 - 48 - 48 = -16$
$D = u_{yy}\,u_{zz} - (uyz)^2$
$= (-24)(-24) - (-16)^2$
$= 576 - 256$
$= 320 > 0$
For $y = 2$ and $z = 2$
We have $x = 5 - 2 - 2 = 1$

Example:

Find all points of the plane x+y+z=5 in the first octant at which $f(x,y,z)=xy^2z^2$ has a maximum value.

$f(x,y,z) = xy^2z^2 = xy^2(5 - x - y)^2$
Since $x+y+z = 5$
$f_x = y^2(5-3x-y)(5-x-y)$,
$f_y = 2xy(5-x-2y)(5-x-y)$
Solving $f_x = 0$, $f_y = 0$, we get
$x=1, y=2, z=2 \therefore x+y+z = 5$

$f_{xx}=-y^2(5-3x-y) - 3y^2(5-x-5)$
$f_{xy}=2y(5-x-y)(5-3x-y)-y^2(5-3x-y)$
$\quad - y^2(5 - x - y)$
$f_{yy}=2x(5-x-y)(5-x-2y)-2xy(5-x-2y)$
$\quad - 4xy(5 - x - y)$
$f_{xx}(1, 2, 2) = -24 < 0$
$f_{yy}(1, 2, 2) = -16$
$f_{xy}(1, 2, 2) = -8$
$f_{xx}f_{yy} - (f_{xy})^2 = (-24)(-16)-(-8)^2$
$= 320 > 0$

Hence "f" has maximum value when x = 1 and y = 2. Thus the points where the function has maximum value is x = 1,y = 2 and z = 2.