Lecture No -16 Extreme Valued Theorem

EXTREME VALUED THEOREM

If the function f is continuous on the closed interval [a, b], then f has an absolute maximum value and an absolute minimum value on [a, b]

Remarks

An absolute extremum of a function on a closed interval must be either a relative extremum or a function value at an end point of the interval. Since a necessary condition for a function to have a relative extremum at a point C is that C be a critical point, we may determine the absolute maximum value and the absolute minimum value of a continuous function f on a closed interval [a, b] by the following procedure.

1. Find the critical points of f on [a, b] and the function values at these critical.
2. Find the values of f (a) and f (b).
3. The largest and the smallest of the above calculated values are the absolute maximum value and the absolute minimum value respectively

Example

Find the absolute extrema of

$$f(x)= x^3+ x^2-x+1 \quad \text{on} \quad [-2,1/2]$$

Since f is continuous on **[-2,1/2],** the extreme value theorem is applicable. For this

$$f'(x) = 3x^2+2x-1$$

This shows that f(x) exists for all real numbers, and so the only critical numbers of **f** will be the values of x for which f (x)=0. .

Setting $f'(x) = 0$, we have

$(3x - 1)(x + 1) = 0$

from which we obtain

$x = -1$ and $x = \frac{1}{3}$

The critical points of f are -1 and $\frac{1}{3}$, and each of these points is in the given closed interval $(-2, \frac{1}{2})$ We find the function values at the critical points and at the end points of the interval, which are given below.

$f(-2) = -1$, $f(-1) = 2$,

$f\left(\frac{1}{3}\right) = \frac{22}{27}$, $f\left(\frac{1}{2}\right) = \frac{7}{8}$

The absolute maximum value of f on $(-2, \frac{1}{2})$ is therefore 2, which occurs at -1, and the absolute min. value of f on $(-2, \frac{1}{2})$ is -1, which occurs at the left end point -2.

Find the absolute extrema of

$$f(x) = (x - 2)^{2/3} \quad \text{on } [1, 5].$$

Since f is continuous on [1. 5], the extreme-value ttheorem is applicable.

Differentiating f with respect to x, we get

$$f'(x) = \frac{2}{3\,(x-2)^{1/3}}$$

There is no value of x for which $f'(x) = 0$. However, since $f'(x)$ does not exist at 2, we conclude that 2 is a critical point of f,

so that the absolute extrema occur either at 2 or at one of the end points of the interval. The function values at these points are given below.

$f(1) = 1$, $f(2) = 0$, $f(5) = \sqrt[3]{9}$

From these values we conclude that the absolute minimum value of f on [1,5] is 0, occurring at 2, and the absolute maximum value of f on [1, 5] is $\sqrt[3]{9}$,occurring at 5.

Find the absolute extrema of

$$h(x) = x^{2/3} \text{ on } [-2, 3].$$

$$h'(x) = \frac{2}{3} x^{-1/3} = \frac{2}{3x^{1/3}}$$

$h'(x)$ has no zeros but is undefined at x = 0.

The values of h at this one critical point and at the endpoints $x = -2$ and $x = 3$ are

$h(0) = 0$

$h(-2) = (-2)^{2/3} = 4^{1/3}$

$h(3) = (3)^{2/3} = 9^{1/3}$.

The absolute maximum value is $9^{1/3}$ assumed at x = 3; the absolute minimum is 0, assumed at x = 0.

How to Find the Absolute Extrema of a Continuous Function f of Two Variables on a Closed and Bounded Region R.

Step 1.

Find the critical points of f that lie in the interior of R.

Setp 2.

Find all boundary points at which the absolute extrema can occur,

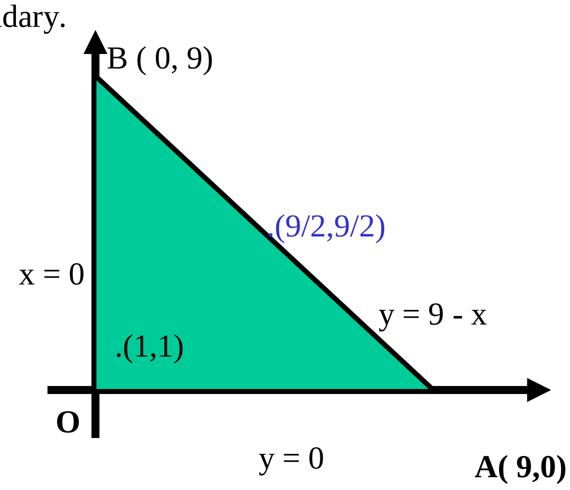
Step 3.

Evaluate f(x,y) at the points obtained in the previous steps. The largest of these values is the absolute maximum and the smallest the absolute minimum.

Find the absolute maximum and minimum value of

$$f(x,y) = 2 + 2x + 2y - x^2 - y^2$$

On the triangular plate in the first quadrant bounded by the lines x=0,y=0,y=9-x
Since f is a differentiable, the only places where f can assume these values are points inside the triangle having vertices at O(0,0), A(9,0)and B(0,9) where $f_x = f_y = 0$ and points of boundary.



<u>For interior points:</u>

We have $f_x=2-2x=0$ and $f_y=2-2y=0$
yielding the single point (1,1)

For boundary points we take take the triangle one side at time :

1. On the segment OA, y=0
   $U(x) = f(x, 0)=2+2x-x^2$

may be regarded as function of x defined on the closed interval $0 \le x \le 9$ Its extreme values may occur at the endpoints x=0 and x=9 which corresponds to points (0, 0) and (9, 0) and U(x) has critical point where

$U'(x) = 2-2x=0$ Then x=1

On the segment OB, x=0 and

$V(y)=f(0,y)= 2+2y-y^2$

Using symmetry of function f, possible points are (0,0),(0,9) and (0,1)

3. The interior points of AB.

With y = 9 - x, we have

$f(x, y) = 2+2x+2(9-x)-x^2-(9-x)^2$

$W(x) = f(x, 9-x) = -61 + 18x - 2x^2$

Setting $w'(x)= 18 - 4x = 0$, $x = 9/2$.

At this value of x, $y = 9 - 9/2$

Therefore we have $\left(\frac{9}{2}, \frac{9}{2}\right)$ as a critical point.

(x, y)	(0,0)	(9,0)	(1, 0)	$\left(\frac{9}{2}, \frac{9}{2}\right)$
f(x,y)	2	-61	3	$-\frac{41}{2}$

(x, y)	(0, 9)	(0,1)	(1,1)
f(x,y)	– 61	3	4

The absolute maximum is 4 which f assumes at the point (1,1) The absolute minimum is -61 which f assumes at the points (0, 9) and (9,0)

EXAMPLE

Find the absolute maximum and the absolute minimum values of

$$f(x,y)=3xy-6x-3y+7$$

on the closed triangular region **r** with the vertices (0,0), (3,0) and (0,5) .

$f(x, y) = 3xy - 6x - 3y + 7$

$f_x(x, y) = 3y - 6, \quad f_y(x, y) = 3x-3$

For critical points

$f_x (x, y) = 0$

$3y - 6 = 0$

$y = 2$

$f_y(x, y) = 0$

$3x - 3 = 0$

$x = 1$

Thus, (1, 2) is the only critical point in the interior of R. Next, we want to determine the location of the points on the boundary of R at which the absolute extrema might occur. The boundary extrema might occur. The boundary each of which we shall treat separately.

i)The line segment between (0, 0) and (3, 0):

On this line segment we have y=0 so (1) simplifies to a function of the single variable x,

$$u(x)=f(x, 0) = -6x + 7, 0 \le x \le 3$$

This function has no critical points because $u^{/}(6)=-6$ is non zero for all x . Thus, the extreme values of u(x) occur at the endpoints x = 0 and x=3 , which corresponds to the points (0, 0) and (3,0) on R

ii) The line segment between the (0,0) and (0,5)

On this line segment we have x=0 ,so single variable y,

$$v(y) = f(0, y) = -3y + 7, \; 0 \le y \le 5$$

This function has no critical points because $v^{/}(y)=-3$ is non zero for all y.Thus ,the extreme values of v(y) occur at the endpoints y = 0 and y=5 which correspond to the point (0,0) and (0,5) or R

iii) The line segment between (3,0) and (0,5)

In the XY- plan , an equation for the line segment

$$y = -\frac{5}{3}x + 5,\ 0 \le x \le 3$$

so (1) simplifies to a function of the single variable x,

$$w(x) = f\left(x, -\frac{5}{3}x + 5\right)$$
$$= -5x^2 + 14x - 8, \quad 0 \le x \le 3$$

$w'(x) = -10x + 14$

$w'(x) = 0$

$10x + 14 = 0$

$x = \frac{7}{5}$

This shows that x =7/5 is the only critical point of w. Thus, the extreme values of w occur either at the critical point x=7/5 or at the endpoints x=0 and x=3.The endpoints correspond to the points (0, 5) and (3, 0) of R, and from (6) the critical point corresponds to [7/5,8/3]

(x, y)	(0, 0)	(3, 0)	(0, 5)	$\left(\frac{7}{5}, \frac{8}{3}\right)$	(1, 2)
f(x,y)	7	– 11	– 8	$-\frac{9}{5}$	1

Finally, table list the values of f(x,y) at the interior critical point and at the points on the boundary where an absolute extremum can occur. From the table we conclude that the absolute maximum value of f is f(0,0)=7 and the absolute minimum values is f(3,0)=-11.

OVER VIEW:

Maxima and Minima of functions of two variables. Page # 833

Exercise: 16.9 Q #26,27,28,29.