Lecture No - 15 Example

EXAMPLE

$$f(x, y) = \sqrt{x^2 + y^2}$$

$$f_x(x, y) = \frac{x}{\sqrt{x^2 + y^2}}$$

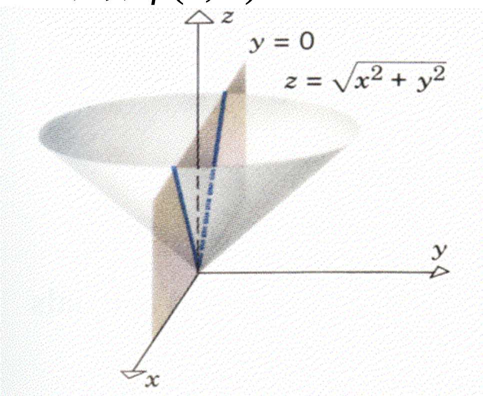
$$f_y(x, y) = \frac{y}{\sqrt{x^2 + y^2}}$$

The partial derivatives exist at all points of the domain of f except at the origin which is in the domain of f. Thus (0, 0) is a critical point of f

Now $f_x(x, y) = 0$ only if $x = 0$ and
$f_y(x, y) = 0$ only if $y = 0$

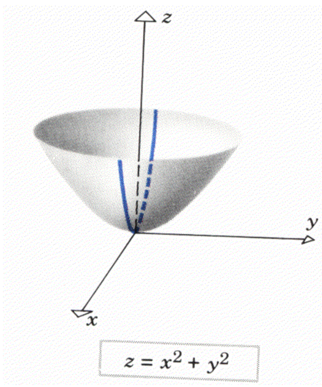
The only critical point is (0,0) and f(0,0)=0

Since $f(x, y) \geq 0$ for all (x, y), $f(0, 0) = 0$ is the absolute minimum value of f.



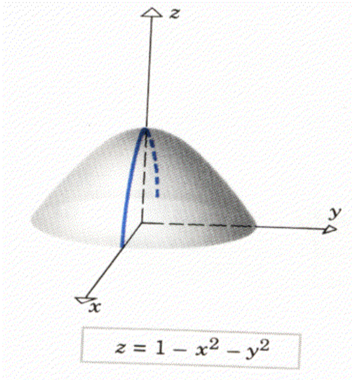
Example

$z = f(x, y) = x^2 + y^2$ (Paraboloid)
$f_x(x, y) = 2x$, $f_y(x, y) = 2y$
when $f_x(x, y) = 0$, $f_y(x, y) = 0$
we have (0, 0) as critical point.



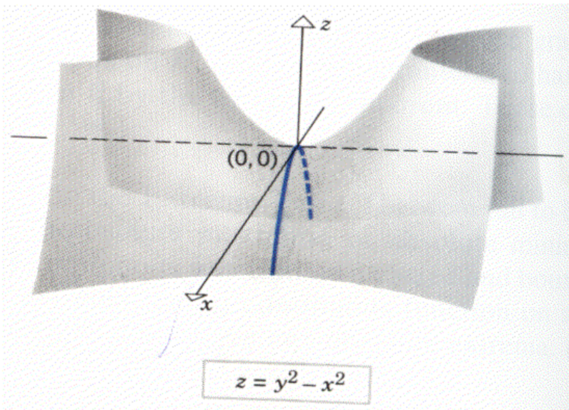
EXAMPLE

$z = g(x, y) = 1 - x^2 - y^2$ (Paraboloid)
$g_x(x, y) = -2x, \quad g_y(x, y) = -2y$
when $g_x(x, y) = 0, \quad g_y(x, y) = 0$
we have (0, 0) as critical point.



EXAMPLE

$z = h(x,y) = y^2 - x^2$ (Hyperbolic paraboloid)
$h_x(x, y) = -2x, \quad h_y(x, y) = 2y$
when $h_x(x, y) = 0, \quad h_y(x, y) = 0$
we have (0, 0) as critical point.



EXAMPLE

$$\mathbf{f(x, y) = \sqrt{x^2 + y^2}}$$

$$f_x = \frac{x}{\sqrt{x^2 + y^2}} \qquad f_y = \frac{y}{\sqrt{x^2 + y^2}}$$

The point (0,0) is critical point of f because the partial derivatives do not both exist. It is evident geometrically that $f_x(0,.0)$ does not exist because the trace of the cone in the plane y=0 has a corner at the origin.

The fact that $f_x(0,0)$ does not exist can also be seen algebraically by noting that $f_x(0,0)$ can be interpreted as the derivative with respect to x of the function

$$f(x, 0) = \sqrt{x^2 + 0} = |x| \quad \text{at } x = 0.$$

But |x| is not differentiable at x = 0, so $f_x(0,0)$ does not exist. Similarly, $f_y(0,0)$ does not exist. The function f has a relative minimum at the critical point (0,0).

The Second Partial Derivative Test

Let *f* be a function of two variables with continuous second order partial derivatives in some circle centered at a critical point (x_0, y_0), and let

$$D = f_{xx}(x_0, y_0)\, f_{yy}(x_0, y_0) - f^2_{xy}(x_0, y_0)$$

(a) If $D > 0$ and $f_{xx}(x_0,y_0) > 0$, then f has a **relative minimum** at (x_0,y_0).

(b) If $D > 0$ and $f_{xx}(x_0,y_0) < 0$, then f has a **relative maximum** at (x_0,y_0).

(c) If $D < 0$, then f has a **saddle point** at (x_0,y_0).

(d) If $D = 0$, then no conclusion can be drawn.

REMARKS

If a function *f* of two variables has an absolute extremum (either an absolute maximum or an absolute minimum) at an interior point of its domain, then this extremum occurs at a critical point.

EXAMPLE

$$f(x,y) = 2x^2 - 4x + xy^2 - 1$$

$f_x(x, y) = 4x - 4 + y^2,\qquad f_{xx}(x, y) = 4$

$f_y(x, y) = 2xy,\qquad f_{yy}(x, y) = 2x$

$f_{xy}(x, y) = f_{yx}(x, y) = 2y$

For critical points, we set the first partial derivatives equal to zero. Then

$$4x - 4 + y^2 = 0 \qquad (1)$$

and

$$2xy = 0 \qquad (2)$$

we have $x = 0$ or $y = 0$

$x = 0$, then from (1), $y = \pm 2$.

$y = 0$, then from (1), $x = 1$.

Thus the critical points are **(1,0), (0, 2), (0, − 2).**

We check the nature of each point.

$f_{xx}(1,0) = 4,$
$f_{yy}(1, 0) = 2,$
$f_{xy}(1, 0) = 0$

$$D = f_{xx}(1, 0).f_{yy}(1, 0) - [f_{xy}(1, 0)]^2$$
$= 8 > 0$

and $f_{xx}(1, 0)$ is positive. Thus f has a relative minimum at (1, 0).

$f_{xx}(0,-2) = 4,$
$f_{yy}(0,-2) = 0,$
$f_{xy}(0,-2) = -4$

$$D = f_{xx}(0, -2).f_{yy}(0, -2) - [f_{xy}(0, -2)]^2$$
$= -16 < 0.$

$f_{xx}(0, 2) = 4,$
$f_{yy}(0, 2) = 0,$
$f_{xy}(0, 2) = 4$

$$D = f_{xx}(0, 2).f_{yy}(0, 2) - [f_{xy}(0, 2)]^2$$
$= -16 < 0.$

Therefore, f has a saddle point at (0,2).

Therefore, f has a saddle point at (0, – 2).

EXAMPLE

$$f(x,y) = e^{-(x^2+y^2+2x)}$$

$$f_x(x, y) = -2(x+1)e^{-(x^2+y^2+2x)},$$
$$f_y(x, y) = -2ye^{-(x^2+y^2+2x)}$$

For critical points

$f_x(x,y) = 0, \quad x + 1 = 0, \quad x = -1$ and

$f_y(x, y) = 0, \quad y = 0$

Hence critical point is (–1,0).

$$f_{xx}(x,y) = [(-2x-2)^2 - 2]e^{-(x^2+y^2+2x)}$$
$f_{xx}(-1, 0) = -$,
$$f_{yy}(x,y) = [4y^2 - 2]e^{-(x^2+y^2+2x)}$$
$f_{yy}(-1, 0) = -$
$$f_{xy}(x,y) = -2y(-2x-2)e^{-(x^2+y^2+2x)}$$
$f_{xy}(-1, 0) = 0$

$$D = f_{xx}(-1,0)\, f_{yy}(-1, 0) - f^2_{xy}(-1, 0)$$
$= (-2e)(-2e) > 0$

This shows that f is maximum at (–1, 0).

EXAMPLE

$$f(x,y) = 2x^4 + y^2 - x^2 - 2y$$

$f_x(x, y) = 8x^3 - 2x$, $\quad f_y(x, y) = 2y - 2$

$f_{xx}(x, y) = 24x^2 - 2$, $\quad f_{yy}(x,y) = 2$,

$f_{xy}(x, y) = 0$

For critical points

$f_x(x, y) = 0$,

$2x(4x^2 - 1) = 0$, $\quad x = 0, 1/2, -1/2$

$f_y(x, y) = 0$,

$2y - 2 = 0$, $\quad y = 1$

Solving above equation we have the critical points $(0,1)$, $\left(-\frac{1}{2}, 1\right)\left(\frac{1}{2}, 1\right)$.

$f_{xx}(0,1) = -2$, $\quad f_{yy}(0, 1) = 2$,

$f_{xy}(0, 1) = 0$

$\mathbf{D = f_x(0, 1)\, f_{yy}(0, 1) - f^2_{xy}(0, 1)}$

$= (-2)(2) - 0 = -4 < 0$

This shows that (0, 1) is a saddle point.

$f_{xx}\left(\frac{1}{2}, 1\right) = 4$, $\quad f_{yy} = \left(\frac{1}{2}, 1\right) = 2$

$f_{xy}\left(\frac{1}{2}, 1\right) = 0$

$\mathbf{D = f_{xx}\left(\frac{1}{2},1\right) f_{yy}\left(\frac{1}{2},1\right) - f^2_{xy}\left(-\frac{1}{2},1\right)}$

$= (4)(2) - 0 = 8 > 0$

$f_{xx}\left(\frac{1}{2},1\right) = 4 > 0$, so f is minimum at $\left(\frac{1}{2},1\right)$.

Example

Locate all relative extrema and saddle points of

$$f(x, y) = 4xy - x^4 - y^4.$$

$f_x(x, y) = 4y - 4x^3$, $f_y(x, y) = 4x - 4y^3$

For critical points

$f_x(x, y) = 0$

$4y - 4x^3 = 0$ (1)

$y = x^3$

$f_y(x, y) = 0$

$4x - 4y^3 = 0$ (2)

$x = y^3$

Solving (1) and (2), we have the critical points (0,0), (1, 1),(−1, −1).

Now $f_{xx}(x, y) = -12x^2$, $f_{xx}(0, 0) = 0$

$f_{yy}(x, y) = -12y^2$, $f_{yy}(0,0) = 0$

$f_{xy}(x, y) = 4$, $f_{xy}(0, 0) = 4$

$\mathbf{D = f_{xx}(0,0)\, f_y(0,0) - f^2_{xy}(0,0)}$

$= (0)(0) - (4)^2 = -16 < 0$

This shows that (0,0) is the saddle point.

$f_{xx}(x, y) = -12x^2$, $f_{xx}(1,1) = -12 < 0$

$f_{yy}(x,y) = -12y^2$, $f_y(1,1) = -12$

$f_{xy}(x, y) = 4$, $f_{xy}(1,1) = 4$

$\mathbf{D = f_{xx}(1,1)\, f_{yy}(1,1) - f^2_{xy}(1, 1)}$

$= (-12)(-12) - (4)^2 = 128 > 0$

This shows that f has relative maximum at (1,1).

$f_{xx}(x,y) = -12x^2$, $f_{xx}(-1, -1) = -12 < 0$

$f_y(x, y) = -2y^2$, $f_y(-1, -1) = -12$

$f_{xy}(x, y) = 4$, $f_{xy}(-1, -1) = 4$

$\mathbf{D = f_{xx}(-1,-1)\, f_{yy}(-1,-1) - f^2_{xy}(-1,-1)}$

$= (-12)(-12) - (4)^2 = 128 > 0$

This shows that f has relative maximum (−1, − 1).

<u>Over view of lecture # </u>15 Book

Calculus by HOWARD ANTON

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