Lecture No -14 Extrema of Functions of Two Variables

In this lecture we shall fined the techniques for finding the highest and lowest points on the graph of a function or , equivalently, the largest and smallest values of the function.

The graph of many functions form hills and valleys. The tops of the hills are relative maxima and the bottom of the valleys are called relative minima. Just as the top of a hill on the earth's terrain need not be the highest point on the earth , so a relative maximum need not be the highest point on the entire graph .

Absolute maximum

A function f of two variables on a subset of $\mathbf{R}^2$ is said to have an D **absolute (global) maximum** value on D if there is some point (x_0, y_0) of D such that value of f on D

$$f(x_0, y_0) \geq f(x, y) \text{ for all } (x, y) \in D$$

In such a case $f(x_0, y_0)$ is the **absolute maximum**

Relative extremum and absolute extremum

If f has a relative maximum or a relative minimum at (x_0, y_0), then we say that f has a relative extremum at (x_0, y_0), and if f has an absolute maximum or absolute minimum at $(x_0 y_0)$, then we say that f has an absolute extremum at (x_0, y_0).

Absolute minimum

A function f of two variables on a subset D of $\mathbf{R}^2$ is said to have an **absolute (global) minimum** value on D if there is some point (x_0, y_0) of D such that

$$f(x_0, y_0) \leq f(x, y) \text{ for all } (x, y) \in D.$$

In such a case $f(x_0, y_0)$ is the **absolute minimum** value of f on D.

Relative (local) maximum

The function f is said to have a relative (local) maximum at some point (x0,y0) of its domain D if there exists an open disc *K centered* at (x0,y0) and of radius r

$$K = \{(x, y) \in \mathbf{R}^2 : (x - x_0)^2 + (y - y_0)^2 < r^2\}$$

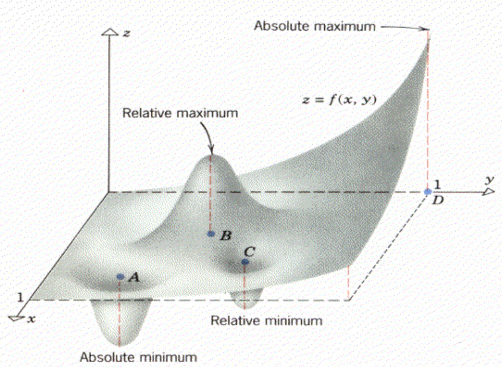
With $K \in D$ such that

$$f(x_0, y_0) \geq f(x, y) \text{ for all } (x, y)$$

Relative (local) minimum

The function f is said to have a **relative(local) minimum** at some point (x_0, y_0) of **D** if there exists an open disc K centred at (x_0, y_0) and of radius r with $K \subset D$ such that

$$f(x_0, y_0) \le f(x, y) \text{ for all } (x, y) \in K.$$



<u>Extreme Value Theorem</u>

If f (x, y) is continuous on a closed and bounded set R, then f has both an absolute maximum and on absolute minimum on R .

<u>Remarks</u>

If any of the conditions the Extreme Value Theorem fail to hold, then there is no guarantee that an absolute maximum or absolute minimum exists on the region R. Thus, a discontinuous function on a closed and bounded set need not have any absolute extrema, and a continuous function on a set that is not closed and bounded also need not have any absolute extrema.

<u>Extreme values or extrema of *f*</u>

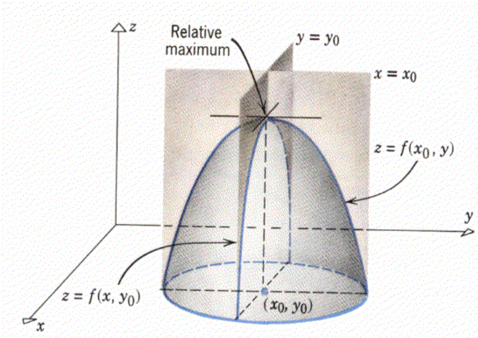
The maximum and minimum values of *f are* referred to as extreme values of extrema of f .Let a function f of two variables be defined on an open disc

$$K = \{(x, y): (x - x_0)^2 + (y - y_0)^2 < r^2\}.$$

Suppose $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ both exist on K

If f has relative extrema at (x_0,y_0),then

$$f_x(x_0, y_0) = 0 = f_y(x_0, y_0).$$



Saddle Point

A differentiable function f(x, y) has a saddle has a saddle point (a, b) if in every open disk centered at (a, b) there are domain points (x, y) where f (x, y) > f (a, b) and domain points (x, y) where f (x, y) < f (a, b). The corresponding point (a, b, f (a, b)) on the surface z = f (x, y) is called a saddle point of the surface

Remarks

Thus, the only points where a function f(x,y) can assume extreme values are critical points and boundary points. As with differentiable functions if a single variable, not every critical point gives rise to o a local extremum. A differentiable function of a single variable might have a point of inflection. A differentiable function of two variable might have a saddle point.

EXAMPLE

Fine the critical points of the given function

$$\boldsymbol{f(x, y) = x^3 + y^3 - 3axy, a > 0.}$$

f_x, f_y exist at all points of the domain of f.

$$f_x = 3x_2 - 3ay, \qquad f_y = 3y^2 - 3ax$$

For critical points $f_x = f_y = 0$.

Therefore, $x^2 - ay = 0 \quad (1)$

and $\quad ax - y^2 = 0 \quad (2)$

Substituting the value of x from (2) into (1),
we have

$$\frac{y^4}{a^2} - ay = 0$$

$$y(y^3 - a^3) = 0$$

$y = 0, \qquad y = a$

and so

$x = 0, \qquad x = a.$

The critical points are (0, 0) and (a, a).

<u>Overview of lecture # 14</u>

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<u>Book</u>

CALCULUS by HOWARD ANTON