Lecture No -13 Orthogonal Surface

In this Lecture we will study the following topics

- **Normal line**
- **Orthogonal Surface**
- **Total differential for function of one variable**
- **Total differential for function of two variables**

Normal line

Let $P_0(x_0,y_0,z_0)$ be any point on the surface $f(x,y,z)=0$ If $f(x,y,z)$ is differentiable at $P_0(x_0,y_0,z_0)$ then the normal line at the point $P_0(x_0,y_0,z_0)$ has the equation

$$x = x_0+f_x(P_0)t, \quad y = y_0+f_y(P_0)t, \qquad z = z_0+f_z(P_0)t$$

Here $\mathbf{f_x}$ means that the function f(x,y,z) is partially differentiable with respect to x And $\mathbf{f_x(P_0)}$ means that the function **f(x,y,z)** is partially differentiable with respect to x at the point $\mathbf{P_0(x_0,y_0,z_0)}$

$\mathbf{f_y}$ means that the function f(x,y,z) is partially differentiable with respect to y And $\mathbf{f_y(P_0)}$ means that the function **f(x,y,z)** is partially differentiable with respect to y at the point $\mathbf{P_0(x_0,y_0,z_0)}$

Similarly

$\mathbf{f_z}$ means that the function f(x,y,z) is partially differentiable with respect to z And $\mathbf{f_z(P_0)}$ means that the function **f(x,y,z)** is partially differentiable with respect to z at the point $\mathbf{P_0(x_0,y_0,z_0)}$

<u>EXAMPLE</u>

Find the Equation of the tangent plane and normal of the surface f(x,y,z)= x2+y2+z2-4 at the point P(1,-2,3)

$$\mathbf{f(x,y,z) = x^2+y^2+z^2 - 14}$$

$$\mathbf{P\ (1, -2, 3).}$$

$$f_x = 2x, \qquad f_y = 2y, \qquad f_z = 2z$$

$$f_x(p_0) = 2, \quad f_y(P_0) = -4, \quad f_z(P_0) = 6$$

Equation of the tangent plane to the surface at P is

$$2(x-1)-4(y+2)+6(z-3)=0$$
$$x-2y+3z-14=0$$

Equations of the normal line of the surface through P are

$$\frac{x-1}{2}=\frac{y+2}{-4}=\frac{z-3}{6}$$
$$\frac{x-1}{1}=\frac{y+2}{-2}=\frac{z-3}{3}$$

EXAMPLE

Find the equation of the tangent plane and normal plane

$$4x^2-y^2+3z^2=10 \quad P(2,-3,1)$$

$$f(x,y,z)=4x^2-y^2+3z^2-10$$
$$f_x=8x, \quad f_y=-2y, \quad f_z=6z$$
$$f_x(P)=16, \quad f_y(P)=6, \quad f_z(P)=6$$

Equations of Tangent Plane to the surface through P is

$$16(x-2)+6(y+3)+6(z-1)=0$$
$$8x+3y+3z=10$$

Equations of the normal line to the surface through P are

$$\frac{x-2}{16}=\frac{y+3}{6}=\frac{z-1}{6}$$
$$\frac{x-2}{8}=\frac{y+3}{3}=\frac{z-1}{3}$$

Example

$$z=\frac{1}{2}x^7y^{-2}$$
$$f(x,y,z)=\frac{1}{2}x^7y^{-2}-z$$
$$f_x=\frac{7}{2}x^6.y^{-2}, \quad f_y=-x^7.y^{-3}, \quad f_z=-1$$
$$f_x(2,4,4)=\frac{7}{2}(2)^6(4)^{-2}=14$$
$$f_y(2,4,4)=-(2)^7(4)^{-3}=-2$$
$$f_z(2,4,4)=-1$$

Equation of Tangent at (2, 4, 4) is given by

$$f_x(2,4,4)(x-2)+f_y(2,4,4)(y-4)+f_z(2,4,4)(z-4)=0$$
$$14(x-2)+(-2)(y-4)-(z-4)=0$$
$$14x-2y-z-16=0$$

The normal line has equations

$$x=2+f_x(2,4,4)t, \quad y=4+f_y(2,4,4)t, \quad z=4+f_z(2,4,4)t$$
$$x=2+14t, \quad y=4-2t, \quad z=4-t$$

ORTHOGONAL SURFACES

Two surfaces are said to be orthogonal at a point of their intersection if their normals at that point are orthogonal. They are Said to intersect orthogonally if they are orthogonal at every point common to them.

CONDITION FOR ORTHOGONAL SURFACES

Let (x, y, z) be any point of intersection of

$$f(x, y, z) = 0 \text{----} \quad (1)$$

$$\text{and} \quad g(x, y, z) = 0 \text{-----} (2)$$

Direction ratios of a line normal to (1) are f_x, f_y, f_z

Similarly, direction rations of a line normal to (2) are g_x, g_y, g_z

The two normal lines are orthogonal if and only if

$$f_x g_x + f_y g_y + f_z g_z = 0$$

EXAMPLE

Show that given two surfaces are orthogonal or not

$f(x,y,z) = x^2 + y^2 + z - 16$ (1)

$g(x,y,z) = x^2 + y^2 - 638$ (2)

Adding (1) and (2)

$x^2 + y^2 = \frac{63}{4}$, $z = \frac{1}{4}$ (3)

$f_x = 2x, \quad f_y = 2y \quad f_z = 1$

$g_x = 2x, \quad g_y = 2y \quad g_z = -63$

$f_x g_x + f_y g_y + f_z g_z$

$= 4(x^2 + y^2) - 63$ using (3)

$f_x g_x + f_y g_y + f_z g_z$

$= 4\left(\frac{63}{4}\right) - 63 = 0$

Since they satisfied the condition of orthogonality so they are orthogonal.

Differentials of a functions

For a function $y = f(x)$

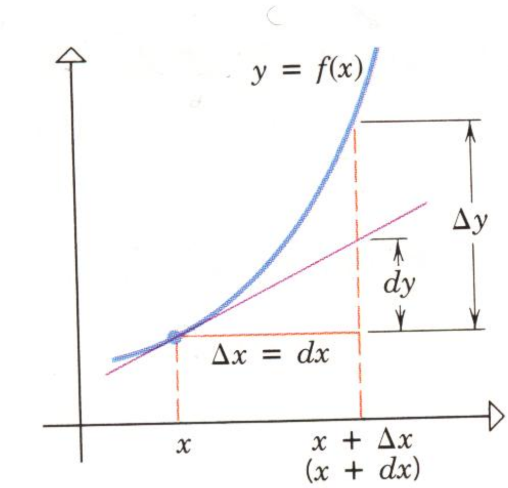
$dy = f'(x)\,dx$

is called the differential of functions f(x)

dx the differential of x is the same as the actual change in x

i.e. $dx = \nabla x$ where as dy the differential of y is the approximate change in the value of the functions which is different from the actual change ∇y in the value of the functions.

Distinction between the increment Δy and the differential dy



Approximation to the curve

If *f* is differentiable at x , then the tangent line to the curve y = f(x) at x_0 is a reasonably good approximation to the curve y = f(x) for value of x near x_0. Since the tangent line passes

through the point $(x_0, f(x_0))$ and has slope $f'(x_0)$, the point-slope form of its equation is

$$y - f'(x_0) = f(x_0)(x - x_0) \quad \text{or}$$

$$y = f(x_0) + f'(x_0)(x - x_0)$$

<u>EXAMPLE</u>

$$\mathbf{f(x) = \sqrt{x}}$$

$$\mathbf{x = 4 \text{ and } dx = \Delta x = 3 \quad y = \sqrt{3}}$$

$$\Delta y = \sqrt{x + \Delta x} - \sqrt{x}$$

$$= \sqrt{7} - \sqrt{4} \approx .65$$

If $y = \sqrt{x}$, then

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} \quad \text{so} \quad dy = \frac{1}{2\sqrt{x}}\, dx$$

$$= \frac{1}{2\sqrt{x}}(3) = \frac{3}{4} = .75$$

EXAMPLE

Using differentials approximation for the value of cos 61 °.

Let y = cosx and x = 60°

then dx = 61° – 60° = 1°

$$\Delta y \approx dy = -\sin x\, dx = -\sin 60°\,(1°)$$

$$= \frac{\sqrt{3}}{2}\left(\frac{1}{180}\pi\right)$$

Now y = cos x

$$y+\Delta y = \cos(x+\Delta x) = \cos(x+dx)$$

$$= \cos(60°+1°) = \cos 61°$$

$$\cos 61° = y+\Delta y = \cos x + \Delta y$$

$$\approx \cos 60° - \frac{\sqrt{3}}{2}\left(\frac{1}{180}\pi\right)$$

$$\cos 61° \approx \frac{1}{2} - \frac{\sqrt{3}}{2}\left(\frac{1}{180}\pi\right)$$

$$= 0.5 - 0.01511 = 0.48489$$

$$\cos 61° \approx 0.48489$$

EXAMPLE

A box with a square base has its height twice is width. If the width of the box is 8.5 inches with a possible error of ± 0.3 inches

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Let x and h be the width and the height of the box respectively, then its volume V is given by

$$V = x^2 h$$

Since h = 2x, so (1) take the form

$$V = 2x^3$$

$$dV = 6x^2\, dx$$

Since x = 8.5, dx = ±0.3, so putting these values in (2), we have

$$dV = 6\,(8.5)^2\,(\pm 0.3) = \pm\, 130.05$$

This shows that the possible error in the volume of the box is ±130.05.

TOTAL DIFFERENTIAL

If we move from (x_0, y_0) to a point $(x_0 + dx, y_0 + dy)$ nearby, the resulting differential in f is

$$df = f_x(x_0, y_0)\, dx + f_y(x_0, y_0)\, dy$$

This change in the linearization of

f is called the total differential of f.

EXACT CHANGE

Area = xy

x = 10, y = 8 Area = 80

x = 10.03 y = 8.02 Area = 80.4406

Exact Change in area = 80.4406 − 80

= 0.4406

EXAMPLE

A rectangular plate expands in such a way that its length changes from 10 to 10.03 and its breadth changes from 8 to 8.02.

Let x and y the length and breadth of the rectangle respectively, then its area is

$$A = xy$$

$$dA = A_x\,dx + A_y dy = ydx + xdy$$

By the given conditions

$$x = 10,\ dx = 0.03,\ y = 8,\ dy = 0.02.$$

$$dA = 8(0.03) + 10(0.02) = 0.44$$

Which is exact Change

EXAMPLE

The volume of a rectangular parallelepiped is given by the formula V = xyz. If this solid is compressed from above so that z is decreased by 2% while x and y each is increased by 0.75% approximately

$$V = xyz$$

$$dV = V_x d\,x + V_y dy + V_z dx$$

$$dV = yzdx + xzdy + zydz \quad (1)$$

$$dx=\frac{0.75}{100}\,x,\ dy = \frac{0.75}{100}\,y,\ dz= -\frac{2}{100}\,z$$

Putting these values in (1), we have

$$dV = \frac{0.75}{100}\,xyz+\frac{0.75}{100}\,xyz - \frac{2}{100}\,xyz$$

$$= -\frac{0.5}{100}\,xyz = -\frac{0.5}{100}\,V$$

This shows that there is 0.5 % decrease in the volume.

EXAMPLE

A formula for the area Δ of a triangle is $\Delta = \frac{1}{2}$ ab sin C. Approximately what error is made in computing Δ if a is taken to be 9.1 instead of 9, b is taken to be 4.08 instead of 4 and C is taken to be $30°3'$ instead of $30°$.

By the given conditions

$$a = 9,\ b = 4,\ C = 30°,$$

$$da = 9.1 - 9 = 0.1,$$

$$db = 4.08 - 4 = 0.08$$

$$dC = 30°3' - 3' = \left(\frac{3}{60}\right)^{\circ}$$

$$= \frac{3}{60} \times \frac{\pi}{180} \text{ radians}$$

Putting these values in (1), we have

$$\Delta = \frac{1}{2} ab \sin C$$

$$d\Delta = \frac{\partial}{\partial a}\left(\frac{1}{2} ab \sin C\right) da + \frac{\partial}{\partial b}\left(\frac{1}{2} ab \sin C\right) db + \frac{\partial}{\partial C}\left(\frac{1}{2} ab \sin C\right) dC$$

$$d\Delta = \frac{1}{2} b \sin C da + \frac{1}{2} a \sin C\, db + \frac{1}{2} ab \cos C dC$$

$$d\Delta = \frac{1}{2} 4 \sin 30°(0.1) + \frac{1}{2} 9 \sin 30°(0.08) + \frac{1}{2} 36 \cos 30° \left(\frac{\pi}{3600}\right)$$

$$d\Delta = 2\left(\frac{1}{2}\right)(0.1) + \frac{1}{2}\left(\frac{1}{2}\right)(0.08) + 18\left(\frac{\sqrt{3}}{2}\right)\left(\frac{3.14}{3600}\right) = 0.293$$

$$\%\text{ change in area} = \frac{0.293}{\Delta} \times 100$$

$$= \frac{0.293}{9} \times 10 = 3.25\%$$