Lecture No -12 Tangent planes to the surfaces

Normal line to the surfaces

If **C** is a smooth parametric curve on three dimensions, then tangent line to **C** at the point P_0 is the line through P_0 along the unit tangent vector to the C at the P_0.The concept of a tangent plane builds on this definition.

If $P_0(x_0,y_0,z_0)$ is a point on the Surface **S,** and if the tangent lines at P_0 to all the smooth curves that pass through P_0 and lies on the surface S all lie in a common plane, then we shall regards that plane to be the ***tangent plane*** to the surface S at P_0.

Its normal (the straight line through P_0 and perpendicular to the tangent) is called the ***surface normal*** of S at P_0.

Different forms of equation of straight line in two dimensional space

1. **Slope intercept form of the Equation of a line.**

$$y = mx + c$$

Where **m** is the slope and **c** is y intercept.

2.Point_Slope Form

Let m be the slope and $P_0(x_0, y_0)$ be the point of required line, then

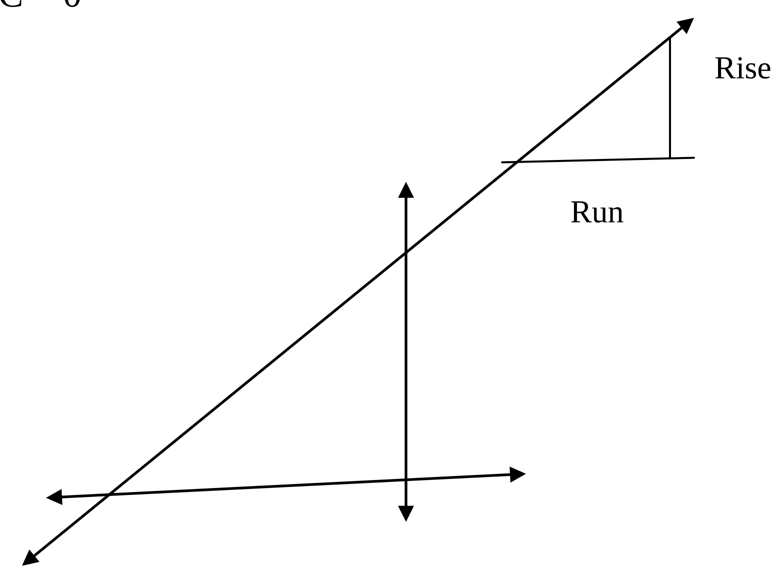
$$y - y_0 = m (x - x_0)$$

3. General Equation of straight line

$$Ax + By + C = 0$$

$$m = \text{slope of line} = \frac{\text{Rise}}{\text{Run}} = \frac{b}{a}$$

$$y - y_0 = \frac{b}{a}(x - x_0)$$



Parametric equation of a line

Parametric equation of a line in two dimensional space passing through the point (x_0, y_0) and parallel to the vector a**i** + b**j** is given by

$x = x_0 + at$, $y = y_0 + bt$

Eliminating t from both equation we get

$$\frac{x - x_0}{a} = \frac{y - y_0}{b}$$

$$y - y_0 = \frac{b}{a}(x - x_0)$$

Parametric vector form:

$$\mathbf{r}(t) = (x_0 + at)\,\mathbf{i} + (y_0 + bt)\,\mathbf{j},$$

Equation of line in three dimensional

Parametric equation of a line in three dimensional space passing through the point (x_0, y_0, z_0) and parallel to the vector ai + bj + ck is given by

$x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$

Eliminating t from these equations we get

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

EXAMPLE

Parametric equations for the straight line through the point A (2, 4, 3) and parallel to the vector **v** = 4**i** + 0**j** – 7**k** .

$x_0 = 2$, $y_0 = 4$, $z_0 = 3$

and a = 4, b = 0 , c = - 7.

The required parametric equations of the straight line are

$x = 2 + 4t$,
$y = 4 + 0t$,
$z = 3 – 7t$

Different form of equation of curve

Curves in the plane are defined in different ways

Explicit form:

$$y = f(x)$$

Example

$$y = \sqrt{9-x^2}\ , \qquad -3 \leq x \leq 3.$$

Implicit form:

$$F(x, y) = 0$$

Example

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$$x^2 + y^2 = 9, \qquad -3 \leq x \leq 3, \qquad 0 \leq y \leq 3$$

Parametric form:

$$x = f(t) \text{ and } y = g(t)$$

Example

$$x = 3\cos\theta, \qquad y = 3\sin\theta, \qquad 0 \leq \theta \leq \pi$$

$$x = 3\cos\theta, \quad y = 3\sin\theta$$

$$x^2 + y^2 = 9\cos^2\theta + 9\sin^2\theta$$

$$= 9(\mathrm{Cos}^2\theta + \mathrm{Sin}^2\theta)$$

$$x^2 + y^2 = 9$$

Parametric vector form:

$$\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j}, \qquad a \leq t \leq b.$$

$$\mathbf{r}(t) = 3\cos\theta\,\mathbf{i} + 3\sin\theta\,\mathbf{j},\ 0 \leq \theta \leq \pi.$$

Equation of a plane

A plane can be completely determined if we know its one point and direction of perpendicular (normal) to it.

Let a plane passing through the point P_0 (x_0, y_0, z_0) and the direction of normal to it is along the vector

$$\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

Let P (x, y, z) be any point on the plane then the line lies on it so that **n** $\perp$ $\overrightarrow{P_0P}$ ($\perp$ means perpendicular to)

$\overrightarrow{P_0P} = (x - x_0)\,\mathbf{i} + (y - y_0)\,\mathbf{j} + (z - z_0)\,\mathbf{k}$

$n \cdot \overrightarrow{P_0P} = 0$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

is the required equation of the plane

Here we use the theorem ,let a and b be two vectors, if a and b are perpendicular then a.b=0 so **n** and P_0P **a**re perpendicular vector so $n.P_0P=0$

REMARKS

Point normal form of equation of plane is

$$\mathbf{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$

We can write this equation as

$$\mathbf{ax + by + cz - ax_0 - by_0 - cz_0 = 0}$$

$$\mathbf{ax + by + cz + d = 0}$$

where $\mathbf{d = -ax_0 - by_0 - cz_0}$

Which is the equation of plane

EXAMPLE

An equation of the plane passing through the point (3, - 1, 7) and perpendicular to the vector **n** = 4**i** + 2**j** - 5**k**.

A point-normal form of the equation is

$$\mathbf{4(x - 3) + 2(y + 1) - 5(z - 7) = 0}$$

$$\mathbf{4x + 2y - 5z + 25 = 0}$$

Which is the same form of the equation of plane **ax + by + cz + d = 0**

The general equation of straight line is $ax + by + c = 0$

Let (x_1, y_1) and (x_2, y_2) be two points on this line then

$ax_1 + by_1 + c = 0$

$ax_2 + by_2 + c = 0$

Subtracting above equation

$a(x_2 - x_1) + b(y_2 - y_1) = 0$

$\mathbf{v} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j}$

is a vector in the direction of line

$\phi(x, y) = ax + by$

$\phi_x = a, \qquad \phi_y = b$

$\nabla\phi = a\mathbf{i} + b\mathbf{j} = \mathbf{n}$

$\nabla\phi \,.\, \mathbf{r} = 0$

Then n and **v** are perpendicular

The general equation of plane is

$ax + by + cz + d = 0$

For any two points (x_1, y_1, z_1) and (x_2, y_2, z_2) lying on this plane we have

$ax_1 + by_1 + cz_1 + d = 0 \qquad (1)$

$a x_2 + by_2 + cz_2 + d = 0 \qquad (2)$

Subtracting equation (1) from (2) have

$a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) = 0$

$(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}).\left[(X_2 - X_1)\, i + (Y_2 - Y_1)\, j + (Z_2 - Z_1)\, k\right]$

Here we use the definition of dot product of two vectors.

$\phi = ax + by + cz$

$\phi_x = a, \qquad \phi_y = b, \qquad \phi_z = c$

$\nabla\phi = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

Where $\mathbf{v} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$

$\nabla\phi$ is always normal to the plane.

Gradients and Tangents to Surfaces

f (x, y) = c

z = f(x,y), z = c

If a differentiable function f(x,y) has a constant value c along a smooth curve having parametric equation

x = g(t), y = h(t), r = g(t)i+h (t)j

differentiating both sides of this equation with respect to t leads to the equation

$$\frac{d}{dt} f\,(g(t), h(t)) = \frac{d}{dt}(\,c\,)$$

$$\frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} = 0 \qquad \text{Chain Rule}$$

$$\left(\frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j}\right).\left(\frac{dg}{dt}i + \frac{dh}{dt}\boldsymbol{j}\right) = 0$$

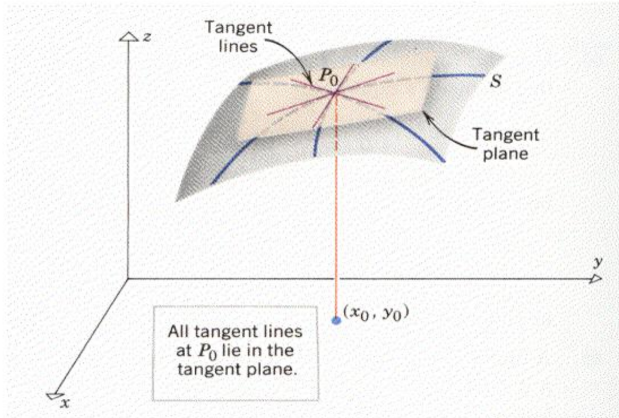
$$\nabla f.\,\frac{d\boldsymbol{r}}{dt} = 0$$

∇f is normal to the tangent vector d **r**/dt,
so it is normal to the curve through (x_0, y_0).

Tangent Plane and Normal Line

Consider all the curves through the point **P0(x0, y0, z0)** on a surface **f(x, y, z) = 0.** The plane containing all the tangents to these curves at the point P0(x0, y0, z0) is called the ***tangent plane to the surface at the point P0***.

The straight lines perpendicular to all these tangent lines at P_0 is called the ***normal line to the surface at P0*** if fx, fy, fz are all continuous at P_0 and not all of them are zero, then gradient *f* (i.e fx**i** + fy**j** + fz**k**) at P0 gives the direction of this normal vector to the surface at P0.



Tangent plane

Let P_0 (x_0, y_0, z_0) be any point on the Surface $f(x,y,z) = 0$. If $f(x,y,z)$ is differentiable at $p_0(x_0,y_0,z_0)$ then the tangents plane at the point P_0 (x_0,y_0,z_0) has the equation

EXAMPLE $9x^2 + 4y^2 - z^2 = 36$ P (2,3,6).

$$f(x,y,z) = 9x^2 + 4y^2 - z^2 - 36$$

$$f_x = 18x, \quad f_y = 8y, \quad f_z = -2z$$

$$f_x(P) = 36, \quad f_y(P) = 24, \quad f_z(P) = -12$$

Equations of Tangent Plane to the surface through P is

$$36(x-2) + 24(y-3) - 12(z-6) = 0$$
$$3x + 2y - z - 6 = 0$$

EXAMPLE

$z = x\cos y - ye^x$ (0,0,0).

$\cos y - ye^x - z = 0$

$f(x,y,z) = \cos y - ye^x - z$

$f_x(0,0,0) = (\cos y - ye^x)_{(0,0)} = 1 - 0.1 = 1$

$f_y(0,0,0) = (-x\sin y - e^x)_{(0,0)} = 0 - 1 = -1.$

$f_z(0,0,0) = -1$

The tangent plane is

$f_x(0,0,0)(x-0) + f_y(0,0,0)(y-0) + f_z(0,0,0)(z-0) = 0$

$1(x-0) - 1(y-0) - 1(z-0) = 0,$

$x - y - z = 0.$