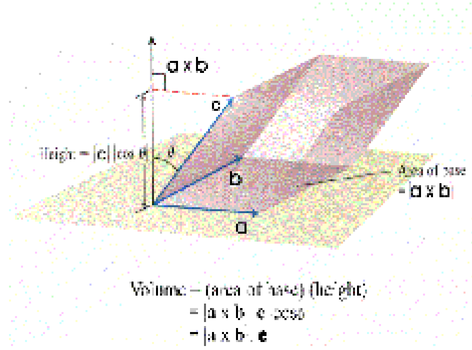
Lecture No -11 The Triple Scalar or Box Product

The product (**a**×**b**) . **c** is called the **triple scalar product** of **a**, **b**, and **c** (in that order).
As | (**a**×**b**).**c** | = |a×**b**| |c| |cos θ|
the absolute value of the product is the volume of the parallelepiped (parallelogram-sided box) determin-ed by a,b and c



By treating the planes of b and c and of c and a as he base planes of the parallelepiped determined by a, b and c

We see that

$$(a * b).c = (b\times c).a=(c\times a).b$$

Since the dot product is commutative, $(a \times b).c = a.(b\times c)$

$$\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$$
$$\mathbf{b} = b_1\,\mathbf{i} + b_2\,\mathbf{j} + b_3\,\mathbf{k}$$
$$\mathbf{c} = c_1\,\mathbf{i} + c_2\,\mathbf{j} + c_3\,\mathbf{k}$$

$$\mathbf{a}.(\mathbf{b}\times\mathbf{c}) = \mathbf{a}.\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \mathbf{a}.\left[\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}\mathbf{k}\right]$$

$$= a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Example

$$\mathbf{a} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}, \quad \mathbf{b} = -2\mathbf{i} + 3\mathbf{k}, \quad \mathbf{c} = 7\mathbf{j} - 4\mathbf{k}.$$

$$\mathbf{a}.(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 3 \\ 7 & -4 \end{vmatrix} - 2\begin{vmatrix} -2 & 3 \\ 0 & -4 \end{vmatrix} - \begin{vmatrix} -2 & 0 \\ 0 & 7 \end{vmatrix}$$

$$= -21 - 16 + 14$$

$$= -23$$

The volume is

$$|\mathbf{a}.(\mathbf{b}\times\mathbf{c})| = 23.$$

When we solve **a.(b×c)** then answer is -23 . if we get negative value then Absolute value make it positive and also volume is always positive.

Gradient of a Scalar Function

$$\nabla \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z},$$

∇ is called "del operator"

Gradient ϕ is a vector operator defined as

$$\text{grad}\,\phi = \left[\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right]\phi$$

$$= \nabla\phi,$$

∇ "del operator" is a vector quantity. Grad means gradient. Gradient is also vector quantity. $\nabla\phi$ is vector and ϕ is scalar quantity, Every component of $\nabla\phi$ will operate with the .

Directional Derivative

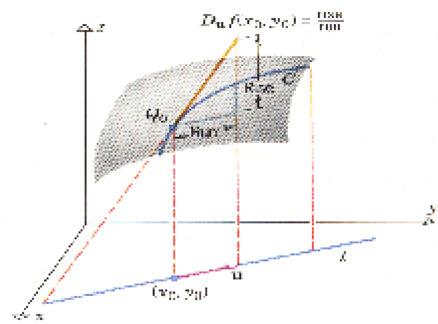
If f(x,y) is differentiable at (x_0,y_0), and if $\mathbf{u} = (u_1, u_2)$ is a unit vector, then the ***directional derivative*** of f at (x_0, y_0) in the direction of **u** is defined by

$D_{\mathbf{u}} f(x_0,y_0) = f_x(x_0,y_0)u_1 + f_y(x_0,y_0)u_2$

It should be kept in mind that there are infinitely many directional derivatives of z = f(x,y) at a point (x_0,y_0), one for each possible choice of the direction vector **u**

Remarks (Geometrical interpretation)

The directional derivative $D_{\mathbf{u}} f(x_0, y_0)$ can be interpreted algebraically as the instantaneous rate of change in the direction of **u** at (x_0, y_0) of $z=f(x,y)$ with respect to the distance parameter s described above, or geometrically as the rise over the run of the tangent line to the curve C at the point Q_0



Example

The directional derivative of $f(x,y) = 3x^2y$ at the point (1, 2) in the direction of the vector a = 3i + 4j.

$$f(x, y) = 3x^2 y$$

$f_x(x, y) = 6xy$, $\quad f_y(x, y) = 3x^2$

so that

$f_x(1, 2) = 12$, $\quad f_y(1, 2) = 3$

$\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{1}{\sqrt{25}}(3\mathbf{i} + 4\mathbf{j})$$

$$= \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$$

$$D_{\mathbf{u}} f(1,2) = 12\left(\frac{3}{5}\right) + 3\left(\frac{4}{5}\right)$$

$$= \frac{48}{5}$$

Note:
Formula for the directional derivative can be written in the following compact form using the gradient notation

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \hat{\mathbf{u}}$$

The dot product of the gradient of *f* with a unit vector $\hat{\mathbf{u}}$ produces the

f_x means that function f(x,y) is differentiating partially with respect to x and f_y means that function f(x,y) is differentiating partially with respect to y.

Example

$$f(x, y) = 2x^2 + y^2, \quad P_0(-1, 1)$$

$$\mathbf{u} = 3\mathbf{i} - 4\mathbf{j}$$

$$|\mathbf{u}| = \sqrt{3^2 + (-4)^2} = 5$$

$$\hat{u} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$$

$f_x = 4x \qquad f_x(-1, 1) = -4$

$f_y = 2y \qquad f_y(-1, 1) = 2$

$$D_{\mathbf{u}} f(-1,1) = f_x(-1,1)u_1 + f_y(-1,1)u_2$$

$$= -\frac{12}{5} - \frac{8}{5} = -4$$

Another example, In this example we have to find directional derivative of the function $f(x, y) = 2x^2 + y^2$ at the point $P_0(-1,1)$ in the direction of **u = 3i – 4j. To find the directional derivative we again use the above formula**

Remarks

If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is a unit vector making an angle θ with the positive x-axis, then

$u_1 = \cos\theta$ and $u_2 = \sin\theta$

$D_{\mathbf{u}} f(x_0,y_0)=f_x(x_0,y_0)u_1 + f_y(x_0,y_0)u_2$

can be written in the form

$D_{\mathbf{u}} f(x_0,y_0) = f_x(x_0,y_0)\cos\theta + f_y(x_0,y_0)\sin\theta$

Example

The directional derivative of e^{xy} at $(-2,0)$ in the direction of the unit vector $\mathbf{u}$ that makes an angle of $\pi/3$ with the positive x-axis.

$$f(x,y) = e^{xy}$$

$f_x(x,y) = ye^{xy}$, $f_y(x,y) = xe^{xy}$

$fx(-2,0) = 0$, $fy(2,) = -2$

$$D_{\mathbf{u}}f(-2,0) = f_x(-2,0)\cos\frac{\pi}{3} + f_y(-2,0)\sin\frac{\pi}{3}$$

$$= 0\left(\frac{1}{2}\right) + (-2)\left(\frac{\sqrt{3}}{2}\right)$$

$$= -\sqrt{3}$$

Gradient of function

If f is a function of x and then the gradient of is defined

$$\nabla f(x,y) = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j}$$

Directional Derivative

Formula for the directional derivative can be written in the following compact form using the gradient

$$D_{\mathbf{u}} f(x,y) = \nabla f(x,y) \cdot \hat{\mathbf{u}}$$

The dot product of the gradient f with a unit $\hat{\mathbf{u}}$ produces directional derivative of f in direction $\mathbf{u}$.

EXAMPLE

$$\mathbf{f(x,y) = 2xy - 3y^2, \quad P_0\ (5,5)}$$

$\mathbf{u} = 4\mathbf{i} + 3\mathbf{j} \quad |\mathbf{u}| = \sqrt{4^2 + 3^2} = 5$

$\hat{u} = \frac{\mathbf{u}}{|\mathbf{u}|} = \frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{j}$

$f_x = 2y, \qquad fy = 2x - by$

$f_x\ (5, 5) = 10, \qquad fy\ (5,5) = -20$

$\nabla f = 10\mathbf{i} - 20\mathbf{j}$

$D_{\mathbf{u}} f(5,5) = \nabla f \cdot \hat{u}$

$= 10\left(\frac{4}{5}\right) - 20\left(\frac{3}{5}\right)$

$= -4$

,In this example we have to find directional derivative of the function $f(x,y) = 2x\ y - 3y^2$ at the point $P_0(5,5)$ in the direction of **u = 4i + 3j. To find the directional derivative we again use the above formula**

EXAMPLE

Directional derivative of the function $f(x, y) = xe^y + \cos(xy)$ at the point (2,0) in the direction of $\mathbf{a} = 3\mathbf{i} - 4\mathbf{j}$.

$\hat{\mathbf{u}} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$

$f_x\ (x,y) = e^y - y \sin(xy)$

$f_y(x,y) = xe^y - x\sin(xy)$

The partial derivatives of f at (2, 0) are

$f_x\ (2,0) = e^0 - 0 = 1$

$f_y(2,0) = 2e^0 - 2.0 = 2$

The gradient of f at (2, 0)

$\nabla f_{(2,0} = {}_x(2, 0)\,\mathbf{i} + {}_y(2, 0)\,\mathbf{j}$

$= \mathbf{i} + \ \mathbf{j}$

The derivative of f at 0) in the direction of a is

$({}_{u}f)_{(2,0)} = \nabla f_{(2,0)} \cdot$

$= \quad \mathbf{j})\left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right)$

$\frac{3}{5} - \frac{8}{5} = -$

Properties of Directional Derivatives

$$\mathbf{D_u f = \nabla f \cdot \hat{u} = |\nabla f| \cos\theta}$$

1. The function f increases most rapidly when $\cos\theta = 1$, or when $\hat{\mathbf{u}}$ is the direction of ∇f. That is, at each point P in its domain, **f increases most rapidly in the direction of the gradient vector ∇ f at P.** The derivative in this direction is

$$D_{\mathbf{u}} f = |\nabla f| \cos(0) = |\nabla f|.$$

2. Similarly, ***f* decreases most rapidly in the direction of $-\nabla$f.**
The derivative in this direction is

$$D_{\mathbf{u}} f = |\nabla f| \cos(\pi) = -|\nabla f|.$$

3. **Any direction $\hat{u}$ orthogonal of the gradient is a direction of zero change in f** because θ equals $\pi/2$ and

$$D_{\mathbf{u}} f = |\nabla f| \cos(\pi/2) = |\nabla f|.0 = 0$$

$$f(\mathbf{x}, \mathbf{y}) = \frac{\mathbf{x}^2}{2} + \frac{\mathbf{y}^2}{2}$$

a) The direction of rapid change

The function increases most rapidly in the direction of ∇f at (1, 1).
The gradient is

$$(\nabla f)_{(1,1)} = (x\mathbf{i} + y\mathbf{j})_{(1,1)} = \mathbf{i} + \mathbf{j}$$

Its direction is

$$\mathbf{u} = \frac{\mathbf{i}+\mathbf{j}}{|\mathbf{i}+\mathbf{j}|}$$

$$= \frac{\mathbf{i}+\mathbf{j}}{\sqrt{1^2+1^2}} = \frac{\mathbf{i}+\mathbf{j}}{\sqrt{2}} = \frac{\mathbf{i}}{\sqrt{2}} + \frac{\mathbf{j}}{\sqrt{2}}$$

b) The directions of zero change

The directions of zero change at

$$\hat{\mathbf{n}} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

and $$-\hat{\mathbf{n}} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$