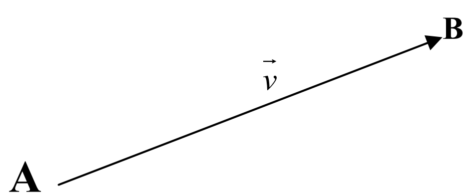
Lecture No. 10

Introduction to Vectors

Some of things we measure are determined by their magnitude, but some times we need magnitude as well as direction to describe the quantities. For example, to describe a force, we need the direction in which that force is acting (Direction) as well as how large it is (Magnitude). Another example is the body's velocity; we have to know where the body is headed as well as how fast it is.

Quantities that have direction as well as magnitude are usually represented by arrows that point the direction of the action and whose lengths give magnitude of the action in term of a suitably chosen unit.

A vector in the plane is a directed line segment.



$$\vec{v} = \overrightarrow{AB}$$

Vectors are usually described by the single **bold** face Roman letters or letter with an arrow. The vector defined by the directed line segment from point A to point B is written as $\overrightarrow{AB}$.

<u>Magnitude or Length of a Vector :</u>

Magnitude of the vector $\vec{v}$ is denoted by

$$|\vec{v}| = |\overrightarrow{AB}|$$

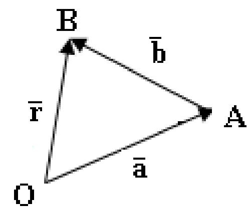
which is the length of the line segment $\overrightarrow{AB}$

<u>Unit vector</u>

Any vector whose magnitude or length is 1 is a unit vector.

Unit vector in the direction of vector $\vec{v}$ is denoted by $\hat{v}$ and is given by $\hat{v} = \dfrac{\vec{v}}{|v|}$

Addition of Vectors



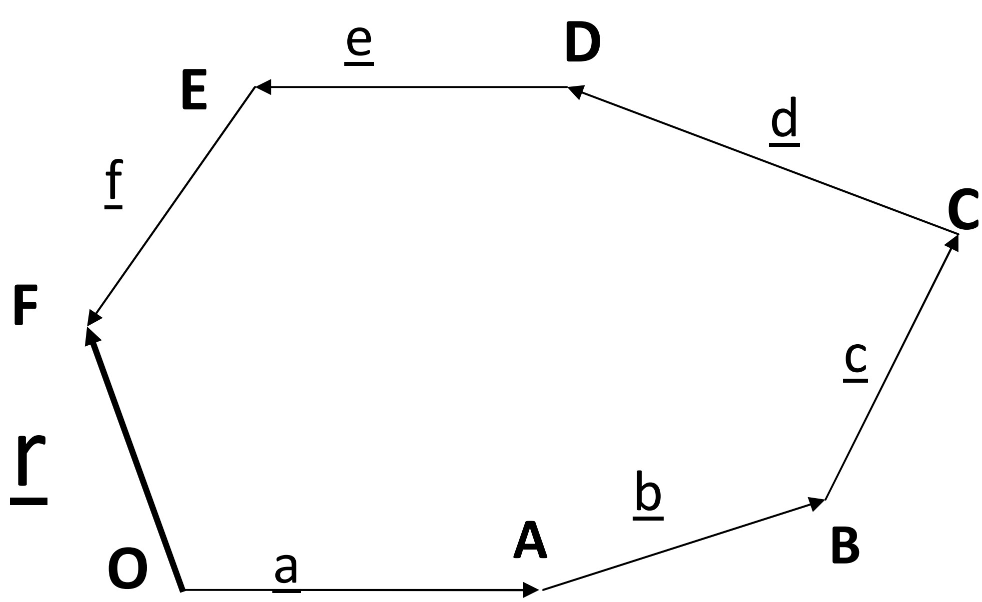
This diagram shows three vectors in two vectors; one vector $\overrightarrow{OA}$ is connected with tail of vector $\overrightarrow{AB}$. The tail of third vector $\overrightarrow{OB}$ is connected with the tail of $\overrightarrow{OA}$ and head is connected with the head of vector $\overrightarrow{AB}$ .This third vector is called Resultant vector $\vec{r}$.

The resultant vector $\vec{r}$ can be written as

$$\vec{r} = \vec{a} + \vec{b}$$

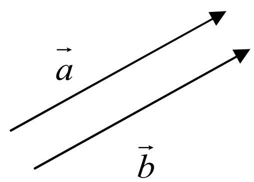
Similarly,

$$\vec{r} = \vec{a} + \vec{b} + \vec{c} + \vec{d} + \vec{e} + \vec{f}$$



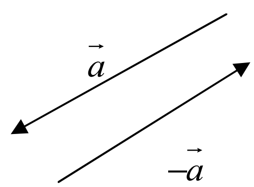
Equal Vectors

Two vectors are equal or same vectors if they have same magnitude and direction. $|\vec{a}| = |\vec{b}|$



Opposite vectors

Two vectors are opposite vectors if they have same magnitude and opposite directions.

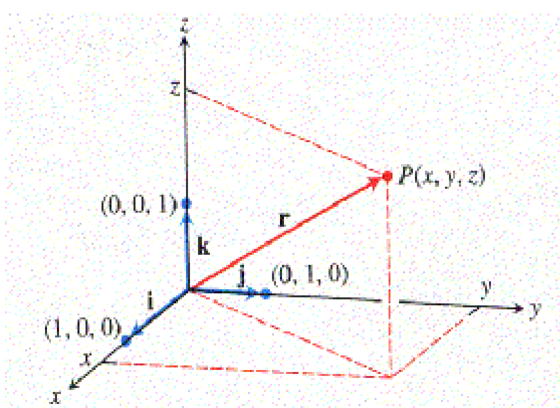


Parallel vectors

Two vectors $\vec{a}$ and $\vec{b}$ are parallel if one vector $\vec{a}$ is scalar multiple of the other $\vec{b}$.

$$\underline{b} = \lambda \underline{a}$$

where λ is a non-zero scalar.



$\underline{r} = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}$

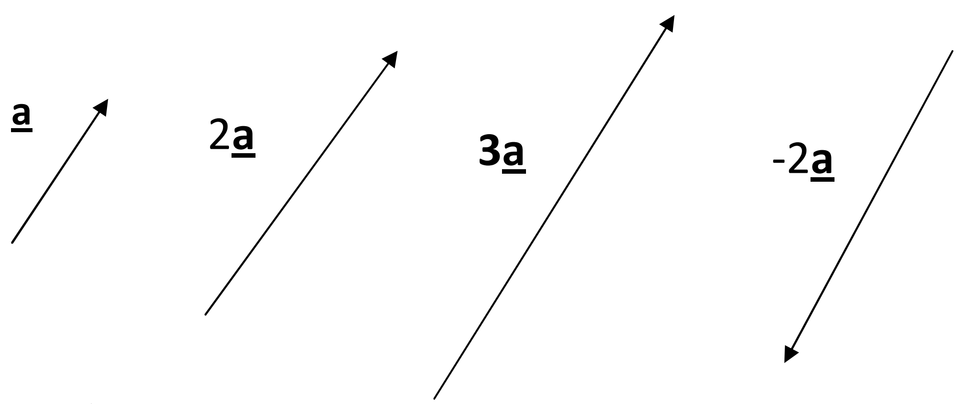
Addition and subtraction of two vectors in rectangular component:

Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$
and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

$$\begin{aligned}\mathbf{a} + \mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) + (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\ &= (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j} + (a_3 + b_3)\mathbf{k} \\ \mathbf{a} - \mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) - (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\ &= (a_1 - b_1)\mathbf{i} + (a_2 - b_2)\mathbf{j} + (a_3 - b_3)\mathbf{k}\end{aligned}$$

The ith component of first vector is added to (or subtracted from) the ith component of second vector, jth component of first vector is added to (or subtracted from) the jth component of second vector, similarly kth component of first vector is added to (or subtracted from) the kth component of second vector.

Multiplication of a vector by a scalar



Any vector $\vec{a}$ can be written as

$$\vec{a} = |\vec{a}|\ \hat{a}$$

Scalar Product

Scalar product (dot product) ("$\vec{a}$ dot $\vec{b}$") of vector $\vec{a}$ and $\vec{b}$ is the number which is given by the formula:

$$\vec{a}.\vec{b} = |\vec{a}|\ |\vec{b}|\ \cos\theta$$

where θ is the angle between $\vec{a}$ and $\vec{b}$.

In words, $\vec{a}.\vec{b}$ is the length of $\vec{a}$ times the length of $\vec{b}$ times the cosine of the angle between $\vec{a}$ and $\vec{b}$.

Remark:-

This is known as commutative law. $\vec{a}.\vec{b} = \vec{b}.\vec{a}$

Some Results of Scalar Product

$$\vec{a}.\vec{b} = |\vec{a}|\ |\vec{b}|\ \cos\theta$$

1) If $\vec{a} \perp \vec{b}$, then it means that $\vec{a}$ is perpendicular to $\vec{b}$

So $\vec{a}.\vec{b} = 0$ $\quad$ $\sin ce\ \theta = 90^0,\ Cos\,90^0 = 0$

Also

$$\hat{i}.\hat{j} = 0 = \hat{j}.\hat{i}$$
$$\hat{j}.\hat{k} = 0 = \hat{k}.\hat{j}$$
$$\hat{k}.\hat{i} = 0 = \hat{i}.\hat{k}$$

2) If $\vec{a} \parallel \vec{b}$ the it means $\vec{a}$ is parallel to $\vec{b}$.

So $\vec{a}.\vec{b} = |\vec{a}|.|\vec{b}|$ $\quad$ since $\theta = 0,\ Cos\,0 = 1$

If we replace $\vec{b}$ by $\vec{a}$, then

$$\vec{a}.\vec{a} = |\vec{a}|.|\vec{a}|$$
$$\vec{a}.\vec{a} = |\vec{a}|^2$$

So $\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$

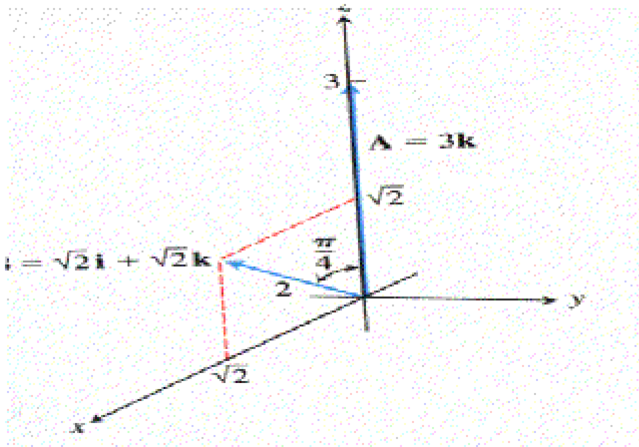
Example

If $\vec{a} = 3\hat{k}$ and $\vec{b} = \sqrt{2}\,\hat{i} + \sqrt{2}\,\hat{k},\quad \theta = \dfrac{\pi}{4},$

then $\vec{a}.\vec{b} = |\vec{a}|\ |\vec{b}|\ \cos\ \theta$

$$= |3\hat{k}|\ |\sqrt{2}\,\hat{i} + \sqrt{2}\,\hat{k}|\ \cos\ \frac{\pi}{4}$$

$$= (3)\ (2)\left(\frac{1}{\sqrt{2}}\right) = \frac{6}{\sqrt{2}} = 3\sqrt{2}$$



EXPRESSION FOR a . b IN COMPONENT FORM

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \quad \text{and}$$
$$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$
$$\begin{aligned}\mathbf{a}.\mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}).(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})\\
&= a_1\mathbf{i}.(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) + a_2\mathbf{j}.(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})\\
&\quad + a_3\mathbf{k}.(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})\\
&= a_1b_1\mathbf{i}.\mathbf{i} + a_1b_2\mathbf{i}.\mathbf{j} + a_1b_3\mathbf{i}.\mathbf{k} + a_2b\mathbf{j}.\mathbf{i} + a_2b_2\mathbf{j}.\mathbf{j}\\
&\quad + a_2b_3\mathbf{j}.\mathbf{k} + a_3b_1\mathbf{k}.\mathbf{i} + + a_3b_2\mathbf{k}.\mathbf{j} + a_3b_3\mathbf{k}.\mathbf{k}\\
&= a_1b_1(1) + a_1b_2(0) + a_1b_3(0) + a_2b_1(0) + a_2b_2(1)\\
&\quad + a_2b_3(0) + a_3b_1(0) + + a_3b_2(0) + a_3b_3(1)\\
&= a_1b_1 + a_2b_2 + a_3b_3\end{aligned}$$

In dot product, the ith component of vector $\vec{a}$ will multiply with ith component of vector $\vec{b}$, jth component of vector $\vec{a}$ will multiply with jth component of vector $\vec{b}$ and kth component of vector $\vec{a}$ will multiply with kth component of vector $\vec{b}$.

Angle between Two Vectors

The angle θ between two vectors $\vec{a}$ and $\vec{b}$ is

$$\theta = Cos^{-1}\left(\frac{\vec{a}.\vec{b}}{\left|\vec{a}\right|.\left|\vec{b}\right|}\right)$$

Since the values of arc-cosine lie in $[0, \pi]$, so the above equation automatically gives the angle made between $\vec{a}$ and $\vec{b}$.

Example

Find the angle between the vectors $\vec{a}=\hat{i}-2\hat{j}-2\hat{k}$ and $\vec{b}=6\hat{i}+3\hat{j}+2\hat{k}$.

Solution:

$$\vec{a}\cdot\vec{b}=\left(\hat{i}-2\hat{j}-2\hat{k}\right).\left(6\hat{i}+3\hat{j}+2\hat{k}\right)$$
$$=(1)(6)+(-2)(3)+(-2)(2)$$
$$=6-6-4=-4$$
$$\left|\vec{a}\right|=\sqrt{(1)^2+(-2)^2+(-2)^2}=\sqrt{9}=3$$
$$\left|\vec{b}\right|=\sqrt{(6)^2+(3)^2+(2)^2}=\sqrt{49}=7$$
$$\theta = Cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{\left|\vec{a}\right|.\left|\vec{b}\right|}\right)=Cos^{-1}\left(\frac{-4}{(3)(7)}\right)=Cos^{-1}\left(\frac{-4}{21}\right)\approx 1.76\ \ radians$$

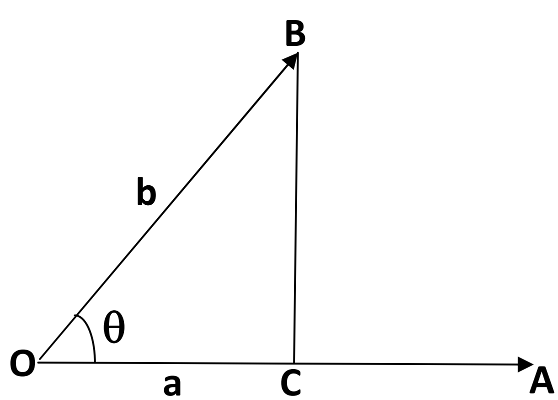
Perpendicular (Orthogonal)Vectors

The non-zero vectors $\vec{a}$ and $\vec{b}$ are perpendicular if and only if $\vec{a}\cdot\vec{b}=0$

This statement has two parts If $\vec{a}$ and $\vec{b}$ are per perpendicular, then $\vec{a}\cdot\vec{b}=0$. And if $\vec{a}\cdot\vec{b}=0$, then $\vec{a}$ and $\vec{b}$ are per perpendicular.

Vector Projection

Consider the Projection of a vector $\vec{b}$ on a vector $\vec{a}$ making an angle θ with each other



From right angle triangle OCB,

$$Cos\,\theta = \frac{base}{hypotenuse}$$

$$Cos\,\theta = \frac{|\overrightarrow{OC}|}{|\vec{b}|}$$

$$|\overrightarrow{OC}| = |\vec{b}|\ Cos\,\theta$$

$$= |\vec{b}|\frac{|\vec{a}|}{|\vec{a}|}Cos\,\theta$$

$$= \frac{\vec{b}.\vec{a}}{|\vec{a}|}$$

Projection of $\vec{b}$ along $\vec{a} = \vec{b}.\frac{\vec{a}}{|\vec{a}|}\left(\frac{\vec{a}}{|\vec{a}|}\right)$ where $\left(\frac{\vec{a}}{|\vec{a}|}\right)$ is the unit vector along $\vec{a}$.

$$= \frac{\vec{b}.\vec{a}}{|\vec{a}||\vec{a}|}\vec{a} = \frac{\vec{b}.\vec{a}}{\vec{a}.\vec{a}}\vec{a}$$

The number $|\vec{b}|\ Cos\,\theta$ is called the scalar component of $\vec{b}$ in the direction of $\vec{a}$ because $|\vec{b}|\ Cos\,\theta = \vec{b}.\hat{a}$

Example

Find the vector projection of $\vec{b} = 6\hat{i} + 3\hat{j} + 2\hat{k}$ onto $\vec{a} = \hat{i} - 2\hat{j} - 2\hat{k}$.

Solution:

Projection of $\vec{b}$ onto $\vec{a} = \frac{\vec{b}.\vec{a}}{\vec{a}.\vec{a}}\vec{a}$

Here,

$$\vec{b}.\vec{a} = \left(6\hat{i} + 3\hat{j} + 2\hat{k}\right).\left(\hat{i} - 2\hat{j} - 2\hat{k}\right) = 6\times1 + 3(-2) + 2(-2) = 6 - 6 - 4 = -4$$

$$\vec{a}.\vec{a} = \left(\hat{i} - 2\hat{j} - 2\hat{k}\right).\left(\hat{i} - 2\hat{j} - 2\hat{k}\right) = 1\times1 + (-2)(-2) + (-2)(-2) = 1 + 4 + 4 = 9$$

$$\text{Projection of } \vec{b} \text{ onto } \vec{a} = \frac{\vec{b}.\vec{a}}{\vec{a}.\vec{a}}\vec{a}$$
$$= \frac{-4}{9}\left(\hat{i} - 2\hat{j} - 2\hat{k}\right)$$
$$= -\frac{4}{9}\hat{i} + \frac{8}{9}\hat{j} + \frac{8}{9}\hat{k}$$

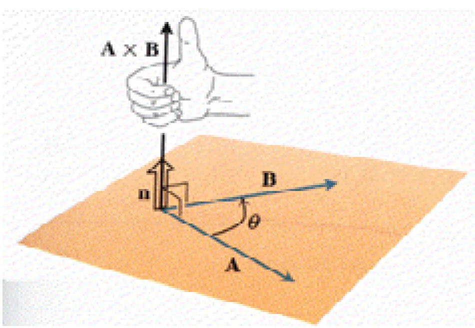
The scalar component of $\vec{b}$ in the direction of $\vec{a}$ is $|\vec{b}|\ Cos\,\theta$.

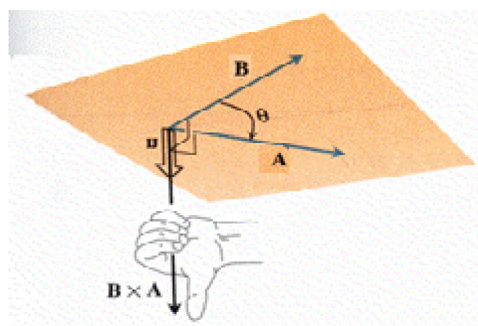
$$|\vec{b}|\ Cos\,\theta = \frac{\vec{b}.\vec{a}}{|\vec{a}|} = \frac{\left(6\hat{i} + 3\hat{j} + 2\hat{k}\right).\left(\hat{i} - 2\hat{j} - 2\hat{k}\right)}{\sqrt{(1)^2 + (-2)^2 + (-2)^2}} = \frac{6\times1 + 3(-2) + 2(-2)}{\sqrt{9}} = \frac{6-6-4}{3} = \frac{-4}{3}$$

The Cross Product of Two Vectors in Space

Consider two non-zero vectors $\vec{a}$ and $\vec{b}$ in space. The vector product $\vec{a}\times\vec{b}$ ("$\vec{a}$ cross $\vec{b}$") to be the vector $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\ Sin\,\theta\ \hat{n}$ where $\hat{n}$ is the unit vector determined by the Right Hand rule.

Right-hand rule





We start with two nonzero nonparallel vectors **A** and **B** .We select a unit vector **n** perpendicular to the plane by the **right handed rule.** This means we choose **n** to be the unit vector that points the way your right thumb points when your fingers curl through the angle 0 from **A** to **B**.
The vector **A** × **B** is orthogonal to both **A** and **B.**

Some Results of Cross Product $\vec{a} \times \vec{b}$

As we know that

$$\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\ Sin\,\theta\ \hat{n}$$

1) If $\vec{a} \| \vec{b}$, then $\vec{a} \times \vec{b} = \vec{0}$ since $Sin\,0^0 = 0$

Similarly,

$$\vec{a} \times \vec{a} = \vec{0}$$

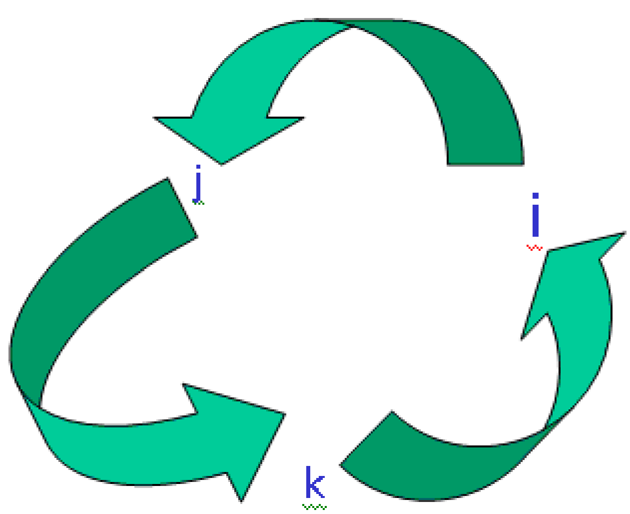
and $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$

2) If $\vec{a} \perp \vec{b}$, then $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\ \hat{n}$ since $Sin\,90^0 = 1$

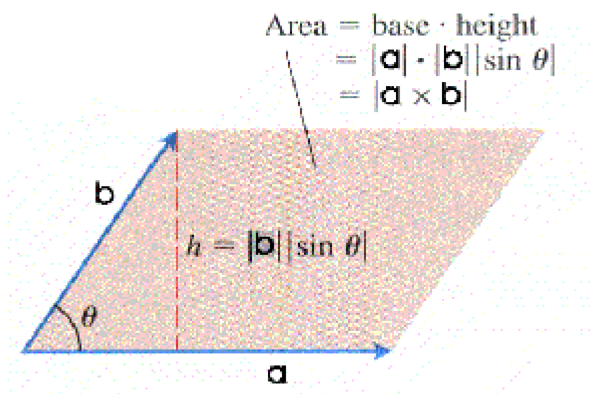
Similarly,

$$\hat{i} \times \hat{j} = \hat{k}, \qquad \hat{j} \times \hat{i} = -\hat{k}$$
$$\hat{j} \times \hat{k} = \hat{i}, \qquad \hat{k} \times \hat{j} = -\hat{i}$$
$$\hat{k} \times \hat{i} = \hat{j}, \qquad \hat{i} \times \hat{k} = -\hat{j}$$

Note that the vector product is not commutative.



The Area of a Parallelogram



Because $\hat{n}$ is a unit vector and magnitude of $\vec{a} \times \vec{b}$ is

$$\left|\vec{a} \times \vec{b}\right| = \left|\vec{a}\right|\left|\vec{b}\right| \, Sin\,\theta \, \left|\hat{n}\right|$$

$$= \left|\vec{a}\right|\left|\vec{b}\right| \, Sin\,\theta \qquad\qquad Since \left|\hat{n}\right| = 1$$

This is the area of parallelogram which is determined by $\vec{a}$ and $\vec{b}$ where $\vec{a}$ is the base and $\left|\vec{b}\right| Sin\,\theta$ is the height of the parallelogram.

a × b from the components of a and b

$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$$\vec{a} \times \vec{b} = \left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right)\times\left(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}\right)$$

$$= a_1\,\hat{i}\times\left(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}\right) + a_2\,\hat{j}\times\left(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}\right) + a_3\,\hat{k}\times\left(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}\right)$$

$$= a_1b_1\hat{i}\times\hat{i} + a_1b_2\hat{i}\times\hat{j} + a_1b_3\hat{i}\times\hat{k} + a_2b_1\hat{j}\times\hat{i} + a_2b_2\hat{j}\times\hat{j} + a_2b_3\hat{j}\times\hat{k}$$
$$a_3b_1\hat{k}\times\hat{i} + a_3b_2\hat{k}\times\hat{j} + a_3b_3\hat{k}\times\hat{k}$$

$$= a_1b_1\times 0 + a_1b_2\hat{k} + a_1b_3\left(-\hat{j}\right) + a_2b_1\left(-\hat{k}\right) + a_2b_2\left(0\right) + a_2b_3\hat{i}$$
$$a_3b_1\hat{j} + a_3b_2\left(-\hat{i}\right) + a_3b_3\left(0\right)$$

$$= (a_2b_3 - a_3b_2)\hat{i} - (a_1b_3 - a_3b_1)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example: Let $\vec{a} = 2\hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = -4\hat{i} + 3\hat{j} + \hat{k}$, then find $\vec{a} \times \vec{b}$.
Solution:

$$\left|\vec{a} \times \vec{b}\right| = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix}$$

$$= \hat{i}(1-3) - \hat{j}(2+4) + \hat{k}(6+4)$$

$$= -2\hat{i} - 6\hat{j} + 10\hat{k}$$

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