Lecture No. 7

Geometric Meaning of Partial Derivative

Geometric meaning of partial derivative

$z = f(x, y)$

Partial derivative of f with respect of x is denoted by $\frac{\partial z}{\partial x}$ *or* f_x *or* $\frac{\partial f}{\partial x}$.

Partial derivative of f with respect of y is denoted by $\frac{\partial z}{\partial y}$ *or* f_y *or* $\frac{\partial f}{\partial y}$.

Partial Derivatives

Let $z = f(x, y)$ be a function of two variables defined on a certain domain D.

For a given change Δx in x, keeping y as constant, the change Δz in z, is given by

$$\Delta z = f(x + \Delta x, y) - f(x, y)$$

If the ratio $\frac{\Delta z}{\Delta x} = \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$ approaches to a finite limit as $\Delta x \to 0$, then this limit is called Partial derivative of f with respect of x.

Similarly for a given change Δy in y, keeping x as constant, the change Δz in z, is given by

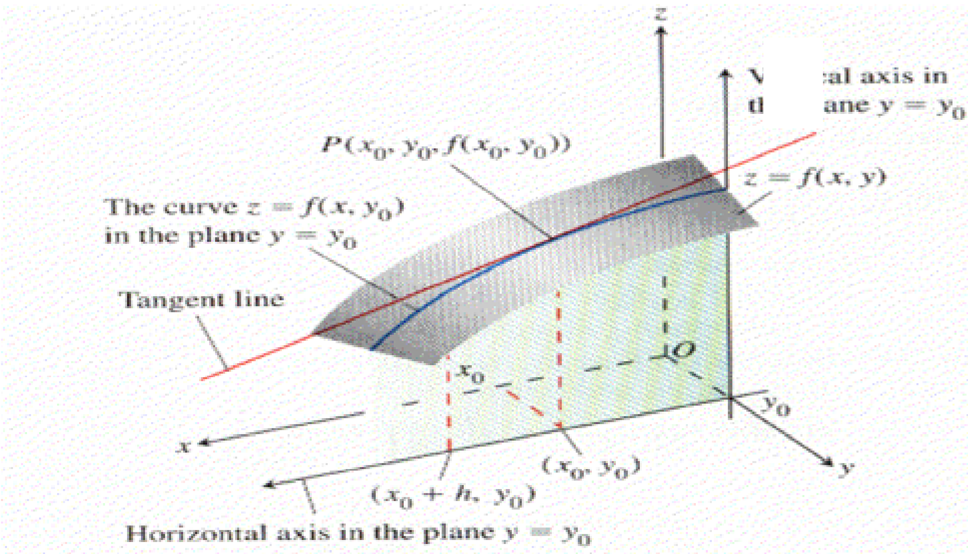
$$\Delta z = f(x, y + \Delta y) - f(x, y)$$

If the ratio $\frac{\Delta z}{\Delta y} = \frac{f(x,\ y + \Delta y) - f(x, y)}{\Delta y}$ approaches to a finite limit as $\Delta y \to 0$, then this limit is called Partial derivative of f with respect of y.

Geometric Meaning of Partial Derivatives

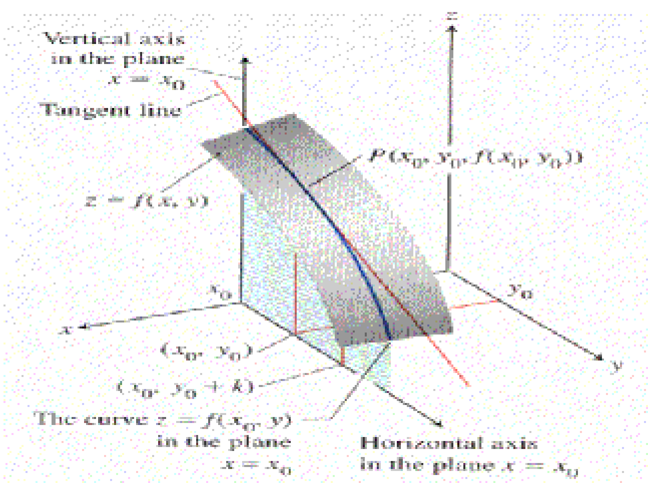
Suppose $z = f(x, y)$ is a function of two variables. The graph of f is a surface. Let P be a point on the graph with the coordinates $(x_0, y_0, f(x_0, y_0))$. If a point starting from P, changes its position on the surface such that y is constant, then the locus of this point is the curve of intersection of $z = f(x, y)$ and y = constant. On this curve, $\frac{\partial z}{\partial x}$ is a derivative of $z = f(x, y)$ with respect to x with y constant.

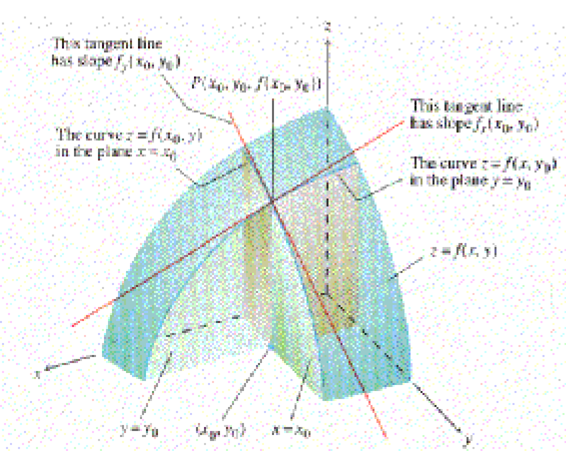
Thus, $\frac{\partial z}{\partial x}$ = slope of the tangent to this curve at P.



Similarly, $\frac{\partial z}{\partial y}$ is the gradient of the tangent at P to this curve of intersection of z = *f(x, y)* and x = constant.

As shown in the figure below (left). Also together these tangent lines are shown in figure below (right).





Partial Derivatives of Higher Orders

The partial derivatives f_x and f_y of a function f of two variables x and y, being functions of x and y, may possess derivatives. In such cases, the second order partial derivatives are defined as below:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}(f_x)=(f_x)_x=f_{xx}=f_{x^2}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial}{\partial y}(f_x)=(f_x)_y=f_{xy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}(f_y)=(f_y)_x=f_{yx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}(f_y)=(f_y)_y=f_{yy}=f_{y^2}$$

Thus there are four second order partial derivatives for a function $z = f(x, y)$. The partial derivatives f_{xy} and f_{yx} are called **Mixed Second partials** and are not equal in general. Partial derivatives of order more than two can be defined in a similar manner.

Example 1: Find $\frac{\partial^2 z}{\partial x\partial y}$ and $\frac{\partial^2 z}{\partial y\partial x}$ for $z = arc\sin\left(\frac{x}{y}\right)$

Solution: $z = arc\sin\left(\frac{x}{y}\right)$

$$\frac{\partial z}{\partial x}=\frac{\partial}{\partial x}\left(arc\sin\left(\frac{x}{y}\right)\right)=\frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\frac{\partial}{\partial x}\left(\frac{x}{y}\right)=\frac{y}{\sqrt{y^2-x^2}}\left(\frac{1}{y}\right)$$

$$=\frac{1}{\sqrt{y^2-x^2}}$$

$$\frac{\partial z}{\partial y}=\frac{\partial}{\partial y}\left(arc\sin\left(\frac{x}{y}\right)\right)=\frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\frac{\partial}{\partial y}\left(\frac{x}{y}\right)=\frac{y}{\sqrt{y^2-x^2}}\left(\frac{-x}{y^2}\right)$$

$$=\frac{-x}{y\sqrt{y^2-x^2}}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{-x}{y\sqrt{y^2-x^2}}\right)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{-1}{y\sqrt{y^2-x^2}} - \frac{x}{y}\left[\frac{x}{\left(y^2-x^2\right)^{\frac{3}{2}}}\right]$$

$$= \frac{-y^2+x^2-x^2}{y\left(y^2-x^2\right)^{\frac{3}{2}}}$$

$$= \frac{-y}{\left(y^2-x^2\right)^{\frac{3}{2}}}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{1}{\sqrt{y^2-x^2}}\right)$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{-1}{2}\left(y^2-x^2\right)^{-\frac{3}{2}} \frac{\partial}{\partial y}\left(y^2-x^2\right)$$

$$= \frac{-1}{2\left(y^2-x^2\right)^{\frac{3}{2}}} \times 2y$$

$$= \frac{-y}{\left(y^2-x^2\right)^{\frac{3}{2}}}$$

Here, you can see that $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$

Example 2

Find $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial^2 f}{\partial x \partial y}$ for $f(x,y) = x\cos y + y\,e^x$.

Solution:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(x\cos y + y\,e^x\right) = \cos y + y\,e^x$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x\cos y + y\,e^x\right) = -x\sin y + e^x$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\cos y + y\,e^x\right) = 0 + y\,e^x = y\,e^x$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(\cos y + y\,e^x\right) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(-x\sin y + e^x\right) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(-x\sin y + e^x\right) = -x\cos y$$

Laplace's Equation

For a function $w = f(x, y, z)$, the equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

is called Laplace's equation.

Example 3: Show that the function $f(x, y) = e^x \sin y + e^y \cos x$ satisfies the Laplace's equation.

Solution: $f(x, y) = e^x \sin y + e^y \cos x$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(e^x \sin y + e^y \cos x\right) = e^x \sin y - e^y \sin x$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(e^x \sin y + e^y \cos x\right) = e^x \cos y + e^y \cos x$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(e^x \sin y - e^y \sin x\right) = e^x \sin y - e^y \cos x$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(e^x \cos y + e^y \cos x\right) = -e^x \sin y + e^y \cos x$$

Adding both partial second order derivatives, we have

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \left(e^x \sin y - e^y \cos x\right) + \left(-e^x \sin y + e^y \cos x\right)$$

$$= 0$$

Euler's Theorem

The Mixed Derivative Theorem

If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy} and f_{yx} are defined throughout an open region containing a point (a, b) and are all continuous at (a, b), then

$$f_{xy}(a\ ,\ b) = f_{yx}\ (a\ ,\ b)$$

Advantage of Euler's theorem

$$w = xy + \frac{e^y}{y^2+1}$$

The symbol $\frac{\partial^2 w}{\partial x \partial y}$ tells us to differentiate first with respect to y and then with respect to x. However, if we postpone the differentiation with respect to y and differentiate first with respect to x, we get the answer more quickly.

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}\left(xy + \frac{e^y}{y^2+1}\right) = y + 0 = y$$

and $\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right) = \frac{\partial}{\partial y}(y) = 1$

Overview of lecture# 7

Chapter # 16 Partial derivatives

Page # 790 Article # 16.3