Lecture No. 38
Taylor and Maclaurin Series

Introduction

When we talk about Approximation, the first question comes in to mind is that, why we have developed these expansion formulas, there is not merely a mathematical curiosity, but rather, is an essential means to exploring and computing those functions (transcendental), whose characteristics are not very much familiar.
What we actually do in approximation problem, we chose a function from the well-defined class that closely matches a target function (which we want to approximate at a certain point) in a task specific way. This is typically done with polynomial or rational (ratio of polynomials) approximations, as we are very well aware of characteristics of polynomials and we know how to mathematically manipulate them to get our required results.
It is common practice to approximate a function by using Taylor series. A Taylor series is a representation of a function as an infinite sum of terms that are calculated from the values of the function's derivatives at a single point. Any finite number of initial terms of the Taylor series of a function is called a Taylor polynomial. The Taylor series of a function is the limit of that function's Taylor polynomials, provided that the limit exists.

Approximation problem

Suppose we are interested in approximating a function $f(x)$ in the neighborhood of a point $a=0$ by a polynomial

$$P(x)=c_0+c_1x+c_2x^2+\cdots+c_nx^n \qquad (1)$$

Because $P(x)$ has $n+1$ coefficients, so we have to impose $n+1$ condition on the polynomial to achieve good approximation to $f(x)$. As "0" is the point about which we are approximating the function so we will chose the coefficient of $P(x)$, such that the $P(x)$ and the 1st n derivatives are same as the $f(x)$ and the 1st n derivatives of $f(x)$ at the point "0" i.e.

$$P(0)=f(0), P'(0)=f'(0),\ P''(0)=f''(0),\dots,P^n(0)=f^n(0) \qquad (2)$$

We have

$$P(x)=c_0+c_1x+c_2x^2+\cdots+c_nx^n$$
$$P'(x)=c_1+2c_2x+\cdots+nc_nx^{n-1}$$
$$P''(x)=2c_2+3.2c_3x+\cdots+n(n-1)c_nx^{n-2}$$
$$P'''(x)=3.2c_3+\cdots+n(n-1)(n-2)c_nx^{n-3}$$
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$$P^n(x)=n(n-1)(n-2)...(1)c_n$$

From (2) we get

$$P(0)=f(0)=c_0$$
$$P'(0)=f'(0)=c_1$$
$$P''(0)=f''(0)=2!c_2$$
$$P'''(0)=f'''(0)=3.2c_3=3!c_3$$
.

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$$P^n(0)=f^n(0)=n(n-1)(n-2)...(1)c_n=n!c_n$$

So we get the following values for the coefficients of

$$P(x)\ c_0=f(0), c_1=f'(0), c_2=\frac{f''(0)}{2!}, c_3=\frac{f'''(0)}{3!}, ..., c_n=\frac{f^n(0)}{n!}$$

Now we have evaluated all the unknowns.

Taylor Polynomial

Let a function f has continuous derivatives of nth order on the interval $[a,a+h]$. Then

$$f(x)=\sum_{k=0}^{n}\frac{f^k(a)}{k!}(x-a)^k = f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\cdots+\frac{(x-a)^n}{n!}f^n(a)$$

Alternate form

$$f(a+h)=\sum_{k=0}^{n}\frac{h^k}{k!}f^k(a)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\cdots+\frac{h^n}{n!}f^n(a)$$

is called the Taylor polynomial of degree n.

Taylor Series

Let a function f has continuous derivatives of every order on the interval $[a,a+h]$. Then

$$f(x)=\sum_{k=0}^{\infty}\frac{f^k(a)}{k!}(x-a)^k=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\cdots+\frac{(x-a)^n}{n!}f^n(a)\ldots$$

Alternate form

$$f(a+h)=\sum_{k=0}^{\infty}\frac{h^k}{k!}f^k(a)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\cdots+\frac{h^n}{n!}f^n(a)+\cdots$$

is called the Taylor Series.

This expression (**Taylor Series)** can be easily converted to **Maclaurin Series** just by putting $a=0$ and $h=x$ the

$$f(x)=\sum_{k=0}^{\infty}\frac{x^k}{k!}f^k(0)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\cdots+\frac{x^n}{n!}f^n(0)+\cdots$$

The above expression is called **Maclaurin Series.**

Taylor's Theorem

Now we will discuss a result called Taylor's Theorem which relates a function, its derivative and its higher derivatives. It basically deals with approximation of functions by polynomials.

Statement

Suppose f has $n+1$ continuous derivatives on an open interval $]a,a+h[$. Then there exist a number $\theta, 0<\theta<1$, such that

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\cdots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+\frac{h^n}{n!}f^n(a+\theta h)$$

Proof:
Consider the function F defined by

$$F(x)=f(x)+(a+h-x)f'(x)+\frac{(a+h-x)^2}{2!}f''(x)+\cdots+\frac{(a+h-x)^{n-1}}{(n-1)!}f^{n-1}(x)$$

$$+\frac{(a+h-x)^n}{n!}A$$

where A is a constant to be determined such that $F(a)=F(a+h)$
So we have

$$f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\cdots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+\frac{h^n}{n!}A=f(a+h)$$

The function F clearly satisfied the condition of roll's Theorem. Hence there exist a number

number θ with $0<\theta<1$, such that , $F'(a+\theta h)=0$

Now

$$F'(x)=f'(x)-f'(x)+(a+h-x)f''(x)-(a+h-x)f''(x)$$

$$+\frac{(a+h-x)^2}{2!}f'''(x)-\cdots+\frac{(a+h-x)^{n-1}}{(n-1)!}f^n(x)-\frac{(a+h-x)^{n-1}}{(n-1)!}A$$

$$=\frac{(a+h-x)^{n-1}}{(n-1)!}[f^n(x)-A]$$

Therefore

$$F'(a+\theta h)=\frac{(h-\theta h)^{n-1}}{(n-1)!}[f^n(a+\theta h)-A]=0$$

$$\frac{h^{n-1}}{(n-1)!}(1-\theta)^{n-1}[f^n(a+\theta h)-A]=0$$

Since $h\neq 0, 1-\theta\neq 0$, so we have

$$f^n(a+\theta h)-A=0$$

$$f^n(a+\theta h)=A$$

Substituting the value of A into (1) we get,

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\cdots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+\frac{h^n}{n!}f^n(a+\theta h)$$

$0<\theta<1$

This formula is known as Taylor Development of a function in finite form with $(n+1)th$ term as **Lagrange's Form of Remainder** after n terms.

Taylor's Theorem with Cauchy Form of Remainder

Suppose *f* has *n*+1 continuous derivatives on an open interval $]a,a+h[$. Then there exist a number θ with $0<\theta<1$, such that

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\cdots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+\frac{h^n}{(n-1)!}(1-\theta)^{n-1}f^n(a+\theta h)$$

This formula is known as Taylor Development of a function in finite form with term as **Cauchy's Form of Remainder** after n terms.

Corollary

If the interval in Taylor's Theorem is taken as $[0,x]$ in place of $[a,a+h]$ then this form is called Maclaurin's Theorem, it again has two forms

I Maclaurin's Theorem with Lagrange's Remainder

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\cdots+\frac{x^{n-1}}{(n-1)!}f^{n-1}(0)+\frac{x^n}{n!}f^n(\theta x)$$

$0<\theta<1$

II Maclaurin's Theorem with Cauchy's Remainder

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\cdots+\frac{x^{n-1}}{(n-1)!}f^{n-1}(0)+\frac{x^n}{(n-1)!}(1-\theta)f^n(\theta x)$$

$0<\theta<1$

Example

Apply Taylor's Theorem to prove that

$$(a+b)^m = a^m + \frac{m}{1!}a^{m-1}b + \frac{m(m-1)}{2!}a^{m-2}b^2 + \cdots$$

For all the real $m, a > 0, -a < b < a$.

Solution

Here $f(x) = x^m$, $f'(x) = mx^{m-1}$, $f''(x) = m(m-1)x^{m-2}$, and

$f'''(x) = m(m-1)(m-2)x^{m-3}, \cdots, f^n(x) = m(m-1)(m-2)\cdots(m-n+1)x^{m-n}$

Then by Taylor's Theorem

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^n}{(n-1)!}(1-\theta)^{n-1}f^n(a+\theta h)$$

By putting values

$$f(x+b) = f(x) + bf'(x) + \frac{b^2}{2!}f''(x) + \frac{b^3}{3!}f'''(x) + \cdots + \frac{b^n}{n!}f^n(x+\theta b)$$

$$= x^m + bmx^{m-1} + \frac{b^2}{2!}m(m-1)x^{m-2} + \frac{b^3}{3!}m(m-1)(m-2)x^{m-3} + \cdots$$

$$+ \frac{b^n}{(n-1)!}m(m-1)(m-2)\cdots(m-n+1)x^{m-n}$$

Then

$$f(a+b) = a^m + bma^{m-1} + \frac{b^2}{2!}m(m-1)a^{m-2} + \frac{b^3}{3!}m(m-1)(m-2)a^{m-3} + \cdots$$

Here

$$R_n = \frac{b^n}{(n-1)!}(1-\theta)^{n-1} f^n(a+\theta b), 0<\theta<1.$$

$$f^n(a+\theta b) = m(m-1)(m-2)\cdots(m-n+1)(a+\theta b)^{m-n}$$

$$R_n = \frac{b^n}{(n-1)!}(1-\theta)^{n-1} m(m-1)(m-2)\cdots(m-n+1)(a+\theta b)^{m-n}$$

$$= \frac{b^n(1-\theta)^{n-1} n!}{(m-n)!(n-1)!}(a+\theta b)^{m-n}$$

$R_n \to 0$ as $n \to \infty$ for all real $m, a>0, -a<b<a$

Hence

$$(a+b)^m = a^m + \frac{m}{1!}a^{m-1}b + \frac{m(m-1)}{2!}a^{m-2}b^2 + \cdots$$

Example

Apply Taylor's Theorem to prove that

$$\ln \sin(x+h) = \ln Sin\, x + hCot\, x - \frac{1}{2}h^2 Csc^2 x + \frac{1}{3}h^3 Cotx Csc^2 x + \cdots$$

Solution

Let $f(x) = \ln Sin\, x,$ then

$$f'(x) = \frac{1}{Sin\, x}.Cosx = Cotx,\ f''(x) = -Csc^2 x$$

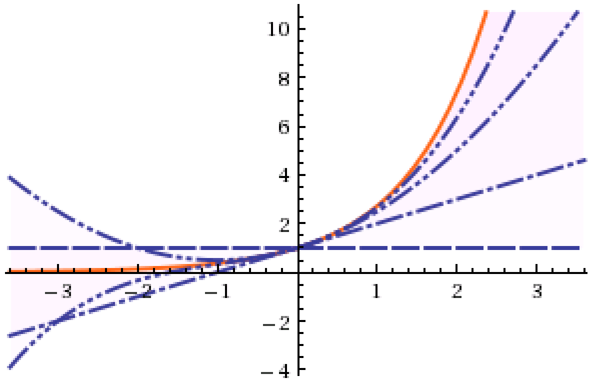
$$f'''(x) = -2Cscx(-Cscx\, Cotx) = 2Csc^2 x\, Cot\, x$$

By applying Taylor's Theorem

$$\ln \sin(x+h) = \ln Sin\, x + hCot\, x - \frac{1}{2}h^2 Csc^2 x + \frac{1}{3}h^3 Cotx Csc^2 x + \cdots$$

Example

Find the Maclaurin Series $f(x) = e^x$, expanded about $x = 0$



Solution

Here $f'(x) = e^x$, $f''(x) = e^x$ and so on $f^{(n)}(x) = e^x$ for $n = 0, 1, 2, \ldots$

$$f(0) = f'(0) = f''(0) = \ldots = f^{(n)}(0) = e^0 = 1$$

The nth Maclaurin polynomial is

$$\sum_{k=0}^{n} \frac{f^k(0)}{k!} x^k = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^n}{n!} f^{(n)}(0)$$

Thus the Maclaurin Series is

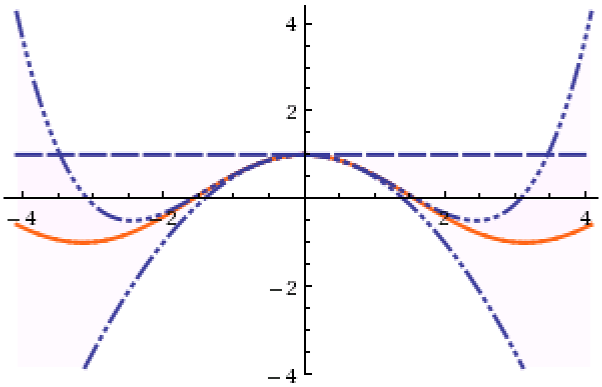
$$\sum_{k=0}^{\infty} \frac{f^k(0)}{k!} x^k = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^n}{n!} f^{(n)}(0) + \cdots$$

Putting the values, we get

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

Example

Find the Maclaurin Series of $f(x) = Cosx$, expanded about $x = 0$.



Solution

Here $f(x) = Cosx$, $f'(x) = -Sinx$, $f''(x) = -Cosx, \ldots$
and $f(0) = 1$, $f'(0) = 0$, $f''(0) = -1, \ldots$

The nth Maclaurin polynomial is

$$\sum_{k=0}^{n} \frac{f^{k}(0)}{k!} x^{k} = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^n}{n!} f^{(n)}(0)$$

the Maclaurin Series is

$$\sum_{k=0}^{\infty} \frac{f^{k}(0)}{k!} x^{k} = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^n}{n!} f^{(n)}(0) + \cdots$$

Putting the values, we get

$$\sum_{k=0}^{\infty} \frac{f^{k}(0)}{k!} x^{k} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + \cdots$$

Exercise

1. Find the Maclaurin series of given functions

i) $Sin\, x$
ii) $e^{Sin\, x}$

2. Find the Taylor series of given functions

i) $\ln x$ about $x = 1$

ii) a^x about $x = 2$