Lecture No. 37

Higher order derivative and Leibniz theorem

Derivative of a function

The concept of **Derivative** is at the core of Calculus and modern mathematics. The definition of the derivative can be approached in two different ways. One is geometrical (as a slope of a curve) and the other one is physical (as a rate of change).

We know that if y=f(x) is a single valued function of a continuous variable, and if the ratio $\frac{1}{h}\{f(x+h)-f(x)\}$ tends to a definite limit as the value of h tends to zero through positive or negative directions, then we say that the function has a derivative at the point 'x' . If the ratio has no limiting value then the function has no derivative at the point x. Symbolically it is represented as

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

If the derivative of a function $y=f(x)$ is itself a continuous function $y'=f'(x)$, we can take the derivative of $f'(x)$, which is generally referred to as the *second derivative of f(x)* and written $f''(x)$. Similarly, the third derivative is obtained by differentiating second derivative as given below.

$$f'''(x)=(f''(x))'$$

This can continue as long as the resulting derivative is itself differentiable, with the fourth derivative, the fifth derivative, and so on.

Any derivative beyond the first derivative can be referred to as a higher order derivative.

Interpretation:

A first derivative tells how fast a function is changing i.e., how fast it's going up or down which is graphically the slope of the curve. A second derivative tells how fast the first derivative is changing or, in other words, how fast the slope is changing. A third derivative informs about how fast the second derivative is changing, i.e., how fast the rate of change of the slope is changing.

Notation

Let $f(x)$ be a function of x. The following are notations for higher order derivatives.

2nd derivative	3rd derivative	4th derivative	nth derivative	remarks
$f''(x)$	$f'''(x)$	$f^{(4)}(x)$	$f^{(n)}(x)$	Probably the most common notation
$\frac{d^2 f}{dx^2}$	$\frac{d^3 f}{dx^3}$	$\frac{d^4 f}{dx^4}$	$\frac{d^n f}{dx^n}$	Leibniz notation.
$\frac{d^2}{dx^2}[f(x)]$	$\frac{d^3}{dx^3}[f(x)]$	$\frac{d^4}{dx^4}[f(x)]$	$\frac{d^n}{dx^n}[f(x)]$	Another form of Leibniz notation.
$D^2 f$	$D^3 f$	$D^4 f$	$D^n f$	Euler's notation.

Because the "prime" notation for derivatives would eventually become somewhat messy, it is preferable to use the numerical notation $f^{(n)}(x) = y^{(n)}(x)$ to denote the nth derivative of f(x).

Example:

$$f(x) = 15x^3 - 3x^2 + 20x - 5$$

Its **first derivative** is given as

$$f'(x) = 45x^2 - 6x + 20$$

Now, this is again a continuous function and therefore can be differentiated. Its derivative which will be the **second derivative** of given function will become

$$f''(x) = (f'(x))' = 90x - 6$$

As, this is a continuous function so we can differentiate it again. This will be called the **third derivative which is**

$$f'''(x) = (f''(x))' = 90$$

Continuing, **fourth derivative** will be

$$f^{(4)}(x) = \left(f'''(x)\right)' = 0$$

(We have changed the notation at this point. We can keep adding on primes, but that will get cumbersome as we calculate the derivatives higher than third.)

This process can continue but notice that we will get zero for all derivatives after this point.
This above example leads us to the following fact about the differentiation of polynomials.

Note:

1) If *p(x)* is a polynomial of degree *n* (*i.e.* the largest exponent in the polynomial) then,
   $$p^{(k)}(x) = 0 \qquad \text{for } k \geq n+1$$
2) We will need to be careful with the "non-prime" notation for derivatives. Consider each of the following

$$f^{(2)}(x) = f''(x)$$
$$f^2(x) = [f(x)]^2$$

The presence of parenthesis in the exponent denotes differentiation while the absence of parenthesis denotes exponentiation.

Example:

If

$f(x) = 3x^4 - 2x^3 + x^2 - 4x + 2$, then

$$f'(x) = 12x^3 - 6x^2 + 2x - 4$$
$$f''(x) = 36x^2 - 12x + 2$$
$$f'''(x) = 72x - 12$$
$$f^{(4)}(x) = 72$$
$$f^{(5)}(x) = 0$$
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$$f^{(n)}(x) = 0 \qquad (n \geq 5)$$

In the above two examples, we have seen that all polynomial functions eventually go to zero when you differentiate repeatedly. On the other hand, rational functions like

$$f(x) = \frac{x^2 - 8}{x + 5}$$

get messier and messier as you take higher and higher derivatives.

Cyclical derivatives:

The higher derivatives of some functions may start repeating themselves. For example, the derivatives of sine and cosine functions behave cyclically.

$$y = \sin x$$
$$y' = \cos x$$
$$y'' = -\sin x$$
$$y''' = -\cos x$$
$$y^{(iv)} = \sin x$$

The cycle repeats indefinitely with every multiple of four.

Example:

Find the third derivative of $f(x) = 4\sin x - \frac{1}{x+3} + 5x$ with respect to x.

Solution:

$$f(x) = 4\sin x - \frac{1}{x+3} + 5x$$

$$f'(x) = 4\cos x + \frac{1}{(x+3)^2} + 5$$

$$f''(x) = -4\sin x - \frac{2}{(x+3)^3} + 0$$

$$f'''(x) = -4\cos x + \frac{6}{(x+3)^4}$$

Some standard nth derivatives

1)

Let

$y = (ax+b)^m$ Then

$$y' = ma(ax+b)^{m-1}$$

$$y'' = m(m-1)a^2(ax+b)^{m-2}$$

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$$y^{(n)} = (m-1)(m-2)......(m-n+1)a^n(ax+b)^{m-n}$$

If m is positive integer and n ≤ m, we can write

$$y^{(n)} = \frac{m!}{(m-n)!}a^n(ax+b)^{m-n}$$

if m=n, then $y^{(n)} = n!a^n$, a constant, so that $y^{(n+1)}$ and subsequent derivatives of y are zero.

Corollary 1:

If $m = -1, y = \frac{1}{ax+b}$

Therefore, $y^{(n)} = (-1)(-2)(-3).....(-n)a^n(ax+b)^{-1-n}$

$$= \frac{(-1)^n n! a^n}{(ax+b)^{n+1}} = \frac{d^n}{dx^n}\left[\frac{1}{ax+b}\right]$$

Corollary 2:

Let $y = \ln(ax+b)$ so that

$$y' = \frac{a}{ax+b} = a.\frac{1}{ax+b}$$

Taking its $(n\text{-}1)th$ derivative, we have

$$y^{(n)} = \frac{d^n}{dx^n}\left[\ln(ax+b)\right] = \frac{d^{n-1}}{dx^{n-1}}\left[\frac{a}{ax+b}\right]$$

$$= a.\frac{(-1)^{n-1}(n-1)!a^{n-1}}{(ax+b)^n} = .\frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$$

2)

$$y = e^{ax}$$

$$y' = ae^{ax}$$

$$y'' = a^2 e^{ax}$$

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$$y^{(n)} = a^{(n)} e^{ax}$$

3)

$$y = \sin(ax+b)$$

$$y' = a\cos(ax+b) = a\sin(ax+b+\frac{\pi}{2})$$

$$y'' = a^2\cos(ax+b+\frac{\pi}{2}) = a^2\sin(ax+b+\frac{\pi}{2}+\frac{\pi}{2}) = a^2\sin(ax+b+2.\frac{\pi}{2})$$

$$y''' = a^3\cos(ax+b+2.\frac{\pi}{2})$$

$$y''' = a^3\sin(ax+b+2.\frac{\pi}{2}+\frac{\pi}{2}) = a^3\sin(ax+b+3.\frac{\pi}{2})$$

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$$y^{(n)} = a^n\sin(ax+b+n.\frac{\pi}{2})$$

Similarly

$$\frac{d^n}{dx^n}[\cos(ax+b)] = a^n \cos(ax+b+n.\frac{\pi}{2})$$

Example:

If $y = \dfrac{x}{2x^2+3x+1}$, find $y^{(n)}$.

Solution:

$$y = \frac{x}{2x^2+3x+1} = \frac{x}{2x^2+2x+x+1} = \frac{x}{2x(x+1)+1(x+1)} = \frac{x}{(2x+1)(x+1)}$$

Applying partial fraction

$$\frac{x}{(2x+1)(x+1)} = \frac{A}{(2x+1)} + \frac{B}{(x+1)} \qquad \text{..................(1)}$$

$$\frac{x}{(2x+1)(x+1)} = \frac{A(x+1)+B(2x+1)}{(2x+1)(x+1)}$$

$$x = A(x+1) + B(2x+1)$$

put $x+1=0 \Rightarrow x=-1$

$$-1 = B(-2+1)$$

$$-1 = -B$$

$$1 = B$$

put $2x+1=0 \Rightarrow x = -\frac{1}{2}$

$$-\frac{1}{2} = A(-\frac{1}{2}+1) = \frac{1}{2}A$$

$$-1 = A$$

put values of A and B in (1)

$$\frac{x}{(2x+1)(x+1)} = \frac{-1}{(2x+1)} + \frac{1}{(x+1)}$$

$$\frac{x}{(2x+1)(x+1)} = \frac{1}{(x+1)} - \frac{1}{(2x+1)}$$

$$y^n = \frac{d^n}{dx^n}\left[\frac{1}{x+1} - \frac{1}{2x+1}\right]$$
$$= \frac{d^n}{dx^n}\left[\frac{1}{x+1}\right] - \frac{d^n}{dx^n}\left[\frac{1}{2x+1}\right]$$
$$= \frac{(-1)^n n!}{(x+1)^{n+1}} - \frac{(-1)^n n! 2^n}{(2x+1)^{n+1}}$$
$$= (-1)^n n!\left[\frac{1}{(x+1)^{n+1}} - \frac{2^n}{(2x+1)^{n+1}}\right]$$

Leibniz theorem

In calculus, the **general Leibniz rule**, named after Gottfried Leibniz, generalizes the product rule (which is also known as "Leibniz's rule".) It states that if *u* and *v* are *n*-times differentiable functions, then the *n*th derivative of the product uv is given by

$$(u.v)^n = \sum_{k=0}^{n} \binom{n}{k} u^{(n-k)} v^{(k)}$$

Where $\binom{n}{k}$ is the binomial coefficient.

Proof:

The proof of this theorem will be given through mathematical induction.

We know that

$$(uv)' = u'v + uv'$$
$$(uv)'' = D^1\left[(uv)'\right]$$
$$= D(u'v + uv')$$
$$= D(u'v) + D(uv')$$
$$= u''v + u'v' + u'v' + uv''$$
$$= u''v + 2u'v' + uv''$$

Which can be written as

$$= {}^2C_0 u''v + {}^2C_1 u'v' + {}^2C_2 uv''.$$

Thus the theorem is true for n=1, 2. Suppose that the theorem is true for a particular value of n, say n=r. then

$$y^{(r)} = (uv)^{(r)} = {}^rC_0\, u^{(r)} v + {}^rC_1\, u^{(r-1)} v' + ... + {}^rC_{r-1}\, u' v^{(r-1)} + {}^rC_r\, u\, v^{(r)}$$

Differentiating both sides of the above equation, we have

$$y^{(r+1)} = (uv)^{(r+1)} = {}^rC_0 [u^{(r+1)} v + u^{(r)} v'] + {}^rC_1 [u^{(r)} v' + u^{(r-1)} v''] + ... + {}^rC_{r-1}[u'' v^{(r-1)} + u' v^{(r)}] + {}^rC_r [u' v^{(r)} + u v^{(r+1)}]$$
$$= {}^rC_0\, u^{(r+1)} v + u^{(r)} v' [{}^rC_0 + {}^rC_1] + u^{(r-1)} v'' [{}^rC_2 + {}^rC_3] + ... + u' v^{(r)} [{}^rC_{r-1} + {}^rC_r] + {}^rC_r u v^{(r+1)}$$

But ${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$ for all n, so that

$$y^{(r+1)} = {}^{r+1}C_0\, u^{(r+1)} v + {}^{r+1}C_1 u^{(r)} v' + {}^{r+1}C_2 u^{(r-1)} v'' + ... + {}^{r+1}C_r u' v^{(r)} + {}^{r+1}C_{r+1} u v^{(r+1)}$$

Thus the theorem is true for n=r+1 .By the principal of mathematical induction, the result is true for all positive integer n. Hence the theorem is proved.

Example:

Find the n^{th} derivative of

$y = e^x lnx$ By using Leibniz theorem

Solution:

Leibniz theorem states that

$$(u.v)^n = \sum_{k=0}^{n} \binom{n}{k} u^{(n-k)}\, v^{(k)}$$

It will be expanded like

$$(u.v)^n = {}^nC_0\, u^{(n)} v + {}^nC_1\, u^{(n-1)} v' + {}^nC_2\, u^{(n-2)} v'' + ... + {}^nC_{n-1}\, u' v^{(n-1)} + {}^nC_n\, u\, v^{(n)}$$

$$(u.v)^n = u^{(n)} v + n u^{(n-1)} v' + \frac{n(n-1)}{2!} u^{(n-2)} v'' + ... + n\, u' v^{(n-1)} + u\, v^{(n)} \qquad (1)$$

Here $u = e^x$ and $v = \ln x$

$u' = e^x \qquad v' = \frac{1}{x} = \frac{(-1)^0 0!}{x}$

$u'' = e^x \qquad v'' = -\frac{1}{x^2} = \frac{(-1)^1 1!}{x^2}$

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$u^{n-1} = e^x \qquad v^{n-1} = \frac{(-1)^{n-2}(n-2)!}{x^{n-1}}$

$u^n = e^x \qquad v^n = \frac{(-1)^{n-1}(n-1)!}{x^n}$

Now inserting all values in (1)

$$(e^x.\ln x)^n = e^x \ln x + ne^x.\frac{1}{x} + \frac{n(n-1)}{2!}e^x(-\frac{1}{x^2}) + ... + ne^x\frac{(-1)^{n-2}(n-2)!}{x^{n-1}} + e^x\frac{(-1)^{n-1}(n-1)!}{x^n}$$

$$= e^x\left[\ln x + \frac{n}{x} + \frac{(-1)n(n-1)}{2x^2} + ... + n\frac{(-1)^{n-2}(n-2)!}{x^{n-1}} + \frac{(-1)^{n-1}(n-1)!}{x^n}\right]$$

Example:

If $y = a\cos(\ln x) + b\sin(\ln x)$, then prove that

$x^2y^{(n+2)} + (2n+1)xy^{(n+1)} + (n^2+1)y^{(n)} = 0$

Solution:

$y = a\cos(\ln x) + b\sin(\ln x)$

$y' = -a\sin(\ln x)\frac{1}{x} + b\cos(\ln x)\frac{1}{x}$

$y' = \frac{1}{x}(-a\sin(\ln x) + b\cos(\ln x))$

$xy' = -a\sin(\ln x) + b\cos(\ln x)$

Differentiating it again, we get

$xy'' + y' = -a\cos(\ln x)\frac{1}{x} - b\sin(\ln x)\frac{1}{x}$

$xy'' + y' = -\frac{1}{x}(a\cos(\ln x) + b\sin(\ln x))$

$x^2y'' + xy' = -(a\cos(\ln x) + b\sin(\ln x)) = -y$

$x^2y'' + xy' + y = 0$

Differentiating 'n' times by using leibniz theorem

$({}^n\mathrm{C}_0\mathrm{y}^{(n+2)}x^2 + {}^n\mathrm{C}_1\mathrm{y}^{(n+1)}.2x + {}^n\mathrm{C}_2\mathrm{y}^{(n)}.2) + ({}^n\mathrm{C}_0\mathrm{y}^{(n+1)}x + {}^n\mathrm{C}_1\mathrm{y}^{(n)}) + y^{(n)} = 0$

$\mathrm{y}^{(n+2)}x^2 + 2xn\mathrm{y}^{(n+1)} + \frac{n(n-1)}{2!}.2.\mathrm{y}^{(n)} + \mathrm{y}^{(n+1)}x + n\mathrm{y}^{(n)} + \mathrm{y}^{(n)} = 0$

$\mathrm{y}^{(n+2)}x^2 + (2n+1)x\mathrm{y}^{(n+1)} + (n^2 - n + n + 1)\mathrm{y}^{(n)} = 0$

$\mathrm{y}^{(n+2)}x^2 + (2n+1)x\mathrm{y}^{(n+1)} + (n^2+1)\mathrm{y}^{(n)} = 0$

$x^2\mathrm{y}^{(n+2)} + (2n+1)x\mathrm{y}^{(n+1)} + (n^2+1)\mathrm{y}^{(n)} = 0$

hence proved.

Example:

Find the nth order derivative of $e^{ax}\sin x$.

Solution:

We know that by using Leibniz theorem

$$(u.v)^n = u^{(n)}v + nu^{(n-1)}v' + \frac{n(n-1)}{2!}u^{(n-2)}v'' + ... + n\,u'v^{(n-1)} + u\,v^{(n)}$$

Here

$u = e^{ax}$ $\quad$ $v = \sin x$

$u' = ae^{ax}$ $\quad$ $v' = \cos x = \sin(x + \frac{\pi}{2})$

$u'' = a^2 e^{ax}$ $\quad$ $v'' = \cos(x + \frac{\pi}{2}) = \sin(x + \frac{\pi}{2} + \frac{\pi}{2}) = \sin(x + 2.\frac{\pi}{2})$

$u''' = a^3 e^{ax}$ $\quad$ $v''' = \cos(x + 2.\frac{\pi}{2}) = \sin(x + 2.\frac{\pi}{2} + \frac{\pi}{2}) = \sin(x + 3.\frac{\pi}{2})$

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$u^{(n-1)} = a^{(n-1)} e^{ax}$ $\quad$ $v^{(n-1)} = \sin(x + (n-1).\frac{\pi}{2})$

$u^{(n)} = a^n e^{ax}$ $\quad$ $v^{(n)} = \sin(x + n.\frac{\pi}{2})$

$$(e^{ax}.\sin x)^{(n)} = a^n e^{ax} \sin x + na^{(n-1)} e^{ax} \sin(x + \frac{\pi}{2}) + \frac{n(n-1)}{2!} a^{(n-2)} e^{ax} \sin(x + 3.\frac{\pi}{2})$$
$$+ .... + nae^{ax} \sin(x + (n-1).\frac{\pi}{2}) + e^{ax} \sin(x + n.\frac{\pi}{2})$$

Exercise

1)
Find the third derivative of $f(x) = 4x^5 + 6x^3 + 2x + 1$ with respect to x.
2)

Find the nth order derivative of

(i) $\dfrac{x}{x^2 - a^2}$

(ii) $\dfrac{x^3}{(x-1)(x-2)}$

3)
Prove that
$$\frac{d^n}{dx^n}\left[\frac{\ln x}{x}\right] = \frac{(-1)^n n!}{x^{n+1}}\left[\ln x - 1 - \frac{1}{2} - ..... - \frac{1}{n-1} - \frac{1}{n}\right]$$
4)
If $f(x) = \ln(1 + \sqrt{1-x})$, prove that
$4x(1-x)f''(x) + 2(2-3x)f'(x) + 1 = 0$
5)

Find the nth order derivative of $e^{ax}\cos x$.