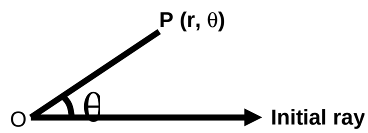
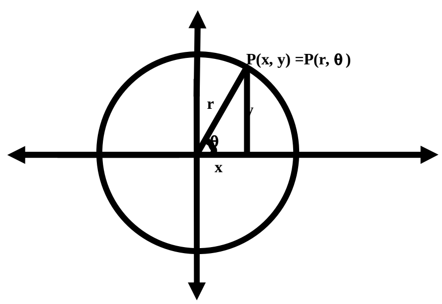
Lecture -4 Polar Coordinates

You know that position of any point in the plane can be obtained by the two perpendicular lines known as x and y axis and together we call it as Cartesian coordinates for plane. Beside this coordinate system we have another coordinate system which can also use for obtaining the position of any point in the plane. In that coordinate system we represent position of each particle in the plane by "r" and "θ "where "r" is the distance from a fixed point known as pole and θ is the measure of the angle.



"O" is known as pole.

Conversion formula from polar to Cartesian coordinates and vice versa



From above diagram and remembering the trigonometric ratios we can write $x = r \cos \theta$, $y = r \sin \theta$. Now squaring these two equations and adding we get,

$$x^2 + y^2 = r^2$$

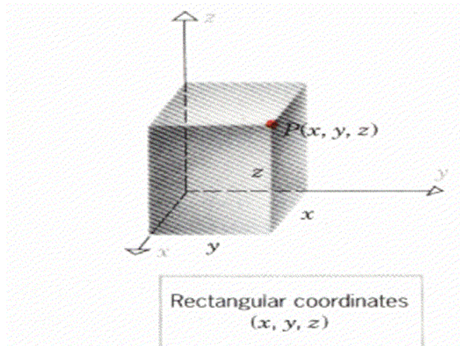
Dividing these equations we get

$$y/x = \tan\theta$$

These two equations gives the relation between the Plane polar and Plane Cartesian coordinates.

Rectangular co-ordinates for 3d

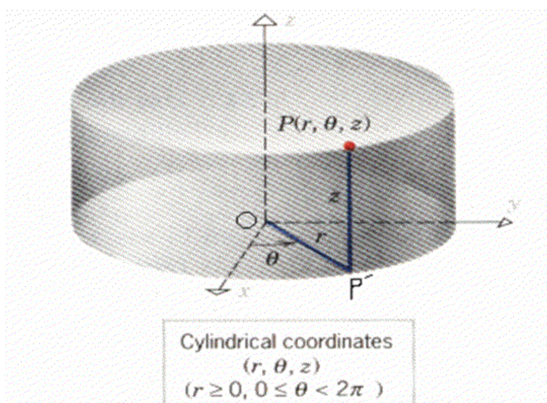
Since you know that the position of any point in the 3d can be obtained by the three mutually perpendicular lines known as x ,y and z – axis and also shown in figure below, these coordinate axis are known as Rectangular coordinate system.



Rectangular coordinates
(x, y, z)

Cylindrical co-ordinates

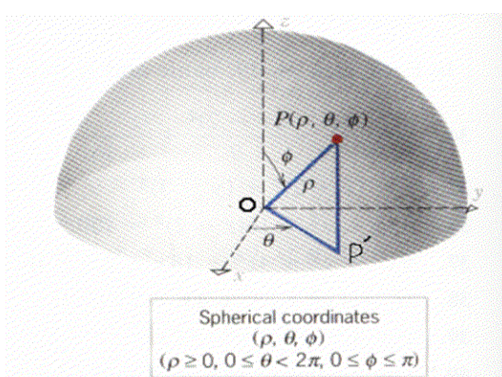
Beside the Rectangular coordinate system we have another coordinate system which is used for getting the position of the any particle is in space known as the cylindrical coordinate system as shown in the figure below.



Cylindrical coordinates
(r, θ, z)
$(r \geq 0, 0 \leq \theta < 2\pi)$

Spherical co-ordinates

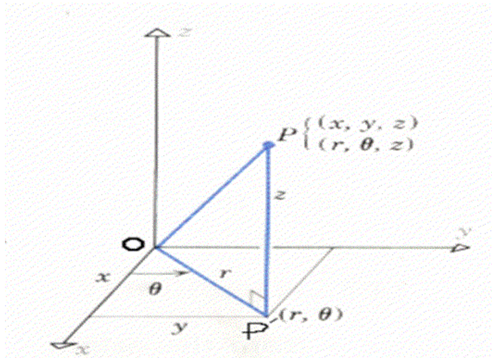
Beside the Rectangular and Cylindrical coordinate systems we have another coordinate system which is used for getting the position of the any particle is in space known as the spherical coordinate system as shown in the figure below.



Spherical coordinates
(ρ, θ, ϕ)
$(\rho \geq 0, 0 \leq \theta < 2\pi, 0 \leq \phi \leq \pi)$

Conversion formulas between rectangular and cylindrical co-ordinates

Now we will find out the relation between the Rectangular coordinate system and Cylindrical coordinates. For this consider any point in the space and consider the position of this point in both the axis as shown in the figure below.



In the figure we have the projection of the point P in the xy-Plane and write its position in plane polar coordinates and also represent the angle θ now from that projection we draw perpendicular to both of the axis and using the trigonometric ratios find out the following relations.

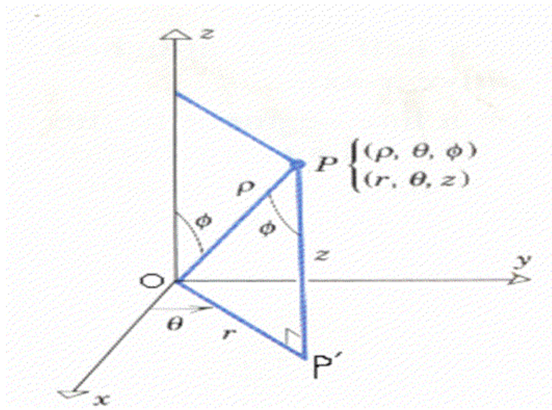
$(r, \theta, z) \rightarrow (x, y, z)$

$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$

$r = \sqrt{x^2+y^2}, \quad \tan\theta = \frac{y}{x}, \quad z = z$

Conversion formulas between cylindrical and spherical co-ordinates

Now we will find out the relation between spherical coordinate system and Cylindrical coordinate system. For this consider any point in the space and consider the position of this point in both the axis as shown in the figure below.



First we will find the relation between Planes polar to spherical, from the above figure you can easily see that from the two right angled triangles we have the following relations.

$(\rho,\theta,\phi) \rightarrow (r, \theta, z)$

$$r = \rho \sin\phi , \quad \theta = \theta, \quad z = \rho \cos \phi$$

Now from these equations we will solve the first and second equation for ρ and ϕ. Thus we have

$$(r, \theta, z) \rightarrow (\rho,\theta,\phi)$$

$$\rho = \sqrt{r^2+z^2} \quad \theta = \theta, \quad \tan \phi = \frac{r}{z}$$

Conversion formulas between rectangular and spherical co-ordinates

$$(\rho, \theta, \Phi) \rightarrow (x, y, z)$$

Since we know that the relation between Cartesian coordinates and Polar coordinates are

$x = r \cos \theta$, $y = r \sin \theta$ and $z = z$ .We also know the relation between Spherical and cylindrical coordinates are,

$$r = \rho \sin\phi , \quad \theta = \theta, \quad z = \rho \cos \phi$$

Now putting this value of "r" and "z" in the above formulas we get the relation between spherical coordinate system and Cartesian coordinate system. Now we will find

$$(x, y, z) \rightarrow (\rho, \theta, \Phi)$$

$$x^2 + y^2 + z^2 = (\rho\sin \Phi \cos \theta)^2 + (\rho\sin \Phi \sin \theta)^2 + (\rho \cos \Phi)^2$$
$$= \rho^2\{\sin^2\Phi(\cos^2\theta + \sin^2\theta) + \cos^2\Phi)\}$$

$$= \rho^2(\sin^2\Phi + \cos^2\Phi)^2 = \rho^2$$

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

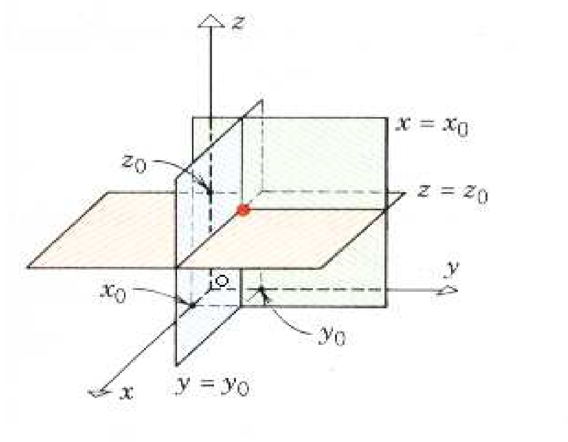
$\tan\theta = y/x$ and $\cos\Phi = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$

Constant surfaces in rectangular co-ordinates

The surfaces represented by equations of the form

$$\mathbf{x = x_0, y = y_0, z = z_0}$$

where x_o, y_o, z_o are constants, are planes parallel to the xy-plane, xz-plane and xy-plane, respectively. Also shown in the figure



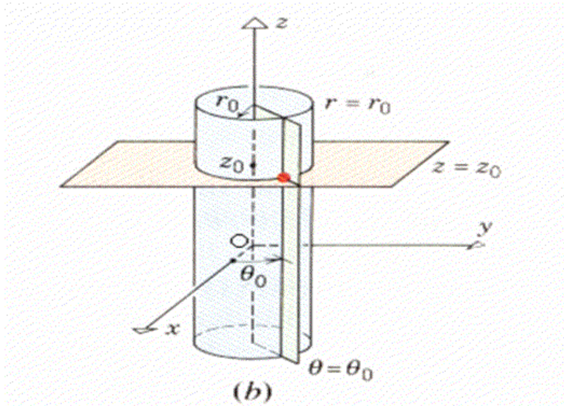
(a)

Constant surfaces in cylindrical co-ordinates

The surface **$r = r_0$ is a right cylinder** of radius r_0 centered on the z-axis. At each point (r, θ, z) this surface on this cylinder, r has the value r_0, z is unrestricted and

$$0 \le \theta < 2\pi.$$

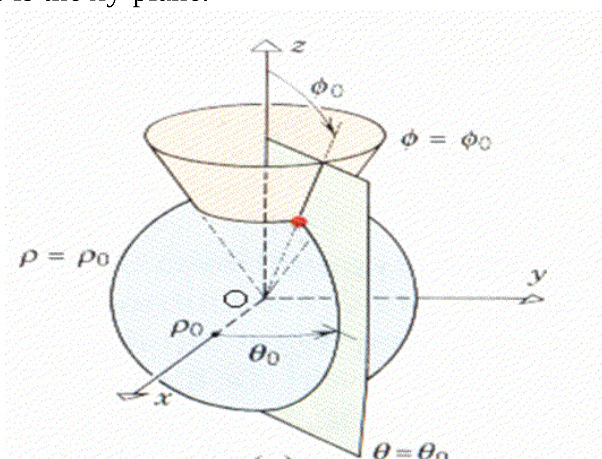
The surface **$\theta = \theta_0$ is a half plane** attached along the z-axis and making angle θ_0 with the positive x-axis. At each point (r, θ, z) on the surface ,θ has the value θ_0, z is unrestricted and $r \geq 0$. The surfaces **$z = z_0$ is a horizontal plane.** At each point (r, θ, z) this surface z has the value z_0 , but r and θ are unrestricted as shown in the figure below.



(b)

Constant surfaces in spherical co-ordinates

The surface **$\rho = \rho_0$ consists of all points whose distance ρ from origin is ρ_0.** Assuming that ρ_0 to be nonnegative, this is a sphere of radius ρ_0 centered at the origin. The surface **$\theta = \theta_0$ is a half plane** attached along the z-axis and making angle θ_0 with the positive x-axis. The surface $\Phi = \Phi_0$ consists of all points from which a line segment to the origin makes an angle of Φ_0 with the positive z-axis. Depending on whether $0 < \Phi_0 < \pi/2$ or $\pi/2 < \Phi_0 < \pi$, this will be a cone opening up or opening down. If $\Phi_0 = \pi/2$, then the cone is flat and the surface is the xy-plane.



(c)

Spherical Co-ordinates in Navigation

Spherical co-ordinates are related to longitude and latitude coordinates used in navigation. Let us consider a right handed rectangular coordinate system with origin at earth's center, positive z-axis passing through the north pole,and x-axis passing through the prime meridian.   Considering earth to be a perfect sphere of radius $\rho = 4000$ miles, then each point has spherical coordinates of the form $(4000,\theta,\Phi)$ where Φ and θ determine the latitude and longitude of the point. Longitude is specified in degree east or west of the prime meridian and latitudes is specified in degree north or south of the equator.

Domain of the Function

- In the above definitions the set D is the domain of the function.
- The Set of all values which the function assigns for every element of the domain is called the Range of the function.
- When the range consist of real numbers the functions are called the real valued function.

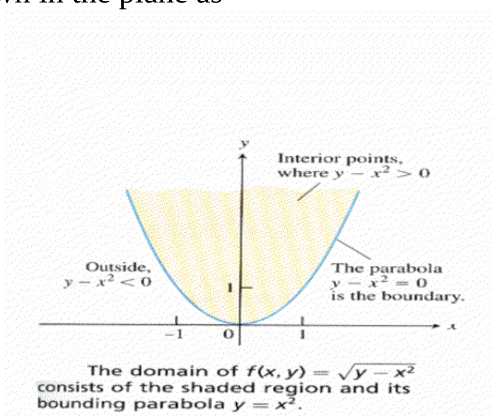
NATURAL DOMAIN

Natural domain consists of all points at which the formula has no divisions by zero and produces only real numbers.

Examples

Consider the Function $\varpi = \sqrt{y - x^2}$. Then the domain of the function is

$y \geq x^2$ Which can be shown in the plane as



and Range of the function is $[0, \infty)$.

Domain of function w = 1/xy is the whole xy- plane Excluding x-axis and y-axis, because at x and y axis all the points has x and y coordinates as 0 and thus the defining formula for the function gives us 1/0. So we exclude them.