61. $y = \sqrt{1 - x^2}$

and E so that the graph of this equation is a circle. What does the graph of $x^2 + y^2 = -9$ look like?

64. *Discussion.* Suppose lighthouse A is located at the origin and lighthouse B is located at coordinates (0, 6). The captain of a ship has determined that the ship's distance from lighthouse A is 2 and its distance from lighthouse B is 5. What are the possible coordinates for the location of the ship?

62. $y = -\sqrt{1 - x^2}$

GRAPHING CALCULATOR EXERCISES

Graph each relation on a graphing calculator by solving for y and graphing two functions.

65. $x^2 + y^2 = 4$

66. $(x - 1)^2 + (y + 2)^2 = 1$

67. $x = y^2$ **68.** $x = (y + 2)^2 - 1$

GETTING MORE INVOLVED

63. *Cooperative learning.* The equation of a circle is a special case of the general equation $Ax^2 + Bx + Cy^2 + Dy = E$, where A, B, C, D, and E are real numbers. Working in small groups, find restrictions that must be placed on A, B, C, D,

69. $x = y^2 + 2y + 1$ **70.** $x = 4y^2 + 4y + 1$

12.4 **THE ELLIPSE AND HYPERBOLA**

In this section we study the remaining two conic sections: the ellipse and the hyperbola.

In this
section

● The Ellipse
● The Hyperbola

The Ellipse

An ellipse can be obtained by intersecting a plane and a cone, as was shown in Fig. 12.3. We can also give a definition of an ellipse in terms of points and distance.

> **Ellipse**
>
> An **ellipse** is the set of all points in a plane such that the sum of their distances from two fixed points is a constant. Each fixed point is called a **focus** (plural: foci).
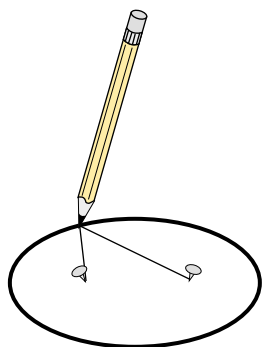
An easy way to draw an ellipse is illustrated in Fig. 12.20. A string is attached at two fixed points, and a pencil is used to take up the slack. As the pencil is moved around the paper, the sum of the distances of the pencil point from the two fixed points remains constant. Of course, the length of the string is that constant. You may wish to try this.
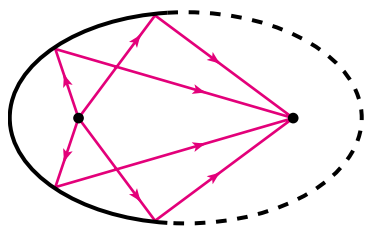
FIGURE 12.20

FIGURE 12.21

Like the parabola, the ellipse also has interesting reflecting properties. All light or sound waves emitted from one focus are reflected off the ellipse to concentrate at the other focus (see Fig. 12.21). This property is used in light fixtures where a concentration of light at a point is desired or in a whispering gallery such as Statuary Hall in the U.S. Capitol Building.

The orbits of the planets around the sun and satellites around the earth are elliptical. For the orbit of the earth around the sun, the sun is at one focus. For the elliptical path of an earth satellite the earth is at one focus and a point in space is the other focus.

Figure 12.22 shows an ellipse with foci $(c, 0)$ and $(-c, 0)$. The origin is the center of this ellipse. In general, the **center** of an ellipse is a point midway between the foci. The ellipse in Fig. 12.22 has x-intercepts at $(a, 0)$ and $(-a, 0)$ and y-intercepts at $(0, b)$ and $(0, -b)$. The distance formula can be used to write the following equation for this ellipse. (See Exercise 55.)
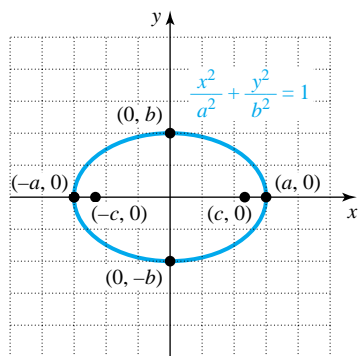

FIGURE 12.22

Equation of an Ellipse Centered at the Origin

An ellipse centered at $(0, 0)$ with foci at $(\pm c, 0)$ and constant sum $2a$ has equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where a, b, and c are positive real numbers with $c^2 = a^2 - b^2$.

To draw a "nice-looking" ellipse, we would locate the foci and use string as shown in Fig. 12.20. We can get a rough sketch of an ellipse centered at the origin by using the x- and y-intercepts only.

EXAMPLE 1

Graphing an ellipse

Find the x- and y-intercepts for the ellipse and sketch its graph.

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

Solution

To find the y-intercepts, let $x = 0$ in the equation:

$$\frac{0}{9} + \frac{y^2}{4} = 1$$
$$\frac{y^2}{4} = 1$$
$$y^2 = 4$$
$$y = \pm 2$$

FIGURE 12.23

To find x-intercepts, let $y = 0$. We get $x = \pm 3$. The four intercepts are $(0, 2)$, $(0, -2)$, $(3, 0)$, and $(-3, 0)$. Plot the intercepts and draw an ellipse through them as in Fig. 12.23. ∎

calculator

close-up

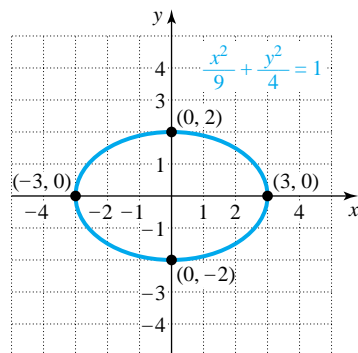To graph the ellipse in Example 1, graph

$$y_1 = \sqrt{4 - 4x^2/9}$$

and

$$y_2 = -\sqrt{4 - 4x^2/9}.$$

Ellipses, like circles, may be centered at any point in the plane. To get the equation of an ellipse centered at (h, k), we replace x by $x - h$ and y by $y - k$ in the equation of the ellipse centered at the origin.

> **helpful hint**
>
> When sketching ellipses or circles by hand, use your hand like a compass and rotate your paper as you draw the curve.

Equation of an Ellipse Centered at (h, k)

An ellipse centered at (h, k) has equation

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1,$$

where a and b are positive real numbers.

EXAMPLE 2

An ellipse with center (h, k)

Sketch the graph of the ellipse:

$$\frac{(x - 1)^2}{9} + \frac{(y + 2)^2}{4} = 1$$

Solution

The graph of this ellipse is exactly the same size and shape as the ellipse

$$\frac{x^2}{9} + \frac{y^2}{4} = 1,$$

which was graphed in Example 1. However, the center for

$$\frac{(x - 1)^2}{9} + \frac{(y + 2)^2}{4} = 1$$

FIGURE 12.24

is $(1, -2)$. The denominator 9 is used to determine that the ellipse passes through points that are three units to the right and three units to the left of the center: $(4, -2)$ and $(-2, -2)$. See Fig. 12.24. The denominator 4 is used to determine that the ellipse passes through points that are two units above and two units below the center: $(1, 0)$ and $(1, -4)$. We draw an ellipse using these four points, just as we did for an ellipse centered at the origin. ∎
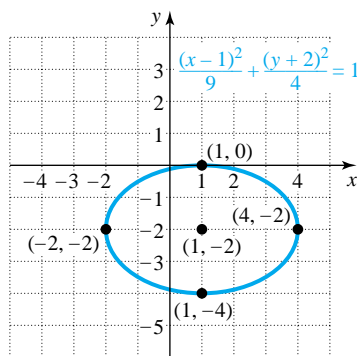
The Hyperbola

A hyperbola is the curve that occurs at the intersection of a cone and a plane, as was shown in Fig. 12.3 in Section 12.2. A hyperbola can also be defined in terms of points and distance.

Hyperbola

A **hyperbola** is the set of all points in the plane such that the difference of their distances from two fixed points (foci) is constant.

Like the parabola and the ellipse, the hyperbola also has reflecting properties. If a light ray is aimed at one focus, it is reflected off the hyperbola and goes to the
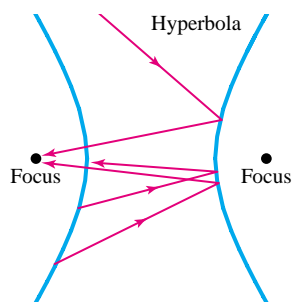
Hyperbola

Focus   Focus

FIGURE 12.25

other focus, as shown in Fig. 12.25. Hyperbolic mirrors are used in conjunction with parabolic mirrors in telescopes.

The definitions of a hyperbola and an ellipse are similar, and so are their equations. However, their graphs are very different. Figure 12.26 shows a hyperbola in which the distance from a point on the hyperbola to the closer focus is N and the distance to the farther focus is M. The value $M - N$ is the same for every point on the hyperbola.
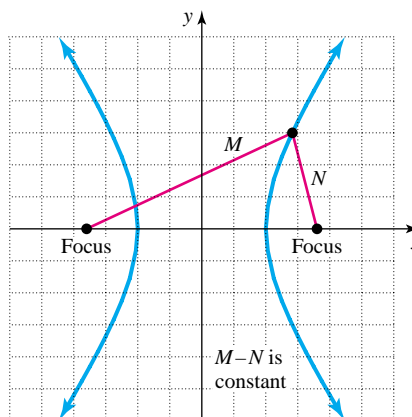

FIGURE 12.26

Fundamental
rectangle   Asymptote

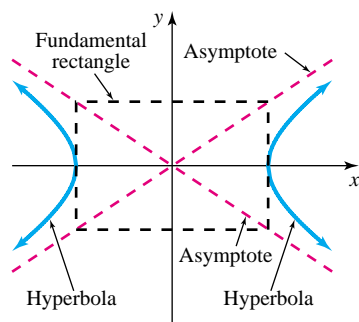Asymptote

Hyperbola   Hyperbola

FIGURE 12.27

A hyperbola has two parts called **branches.** These branches look like parabolas, but they are not parabolas. The branches of the hyperbola shown in Fig. 12.27 get closer and closer to the dashed lines, called **asymptotes,** but they never intersect them. The asymptotes are used as guidelines in sketching a hyperbola. The asymptotes are found by extending the diagonals of the **fundamental rectangle,** shown in Fig. 12.27. The key to drawing a hyperbola is getting the fundamental rectangle and extending its diagonals to get the asymptotes. You will learn how to find the fundamental rectangle from the equation of a hyperbola. The hyperbola in Fig. 12.27 opens to the left and right.

If we start with foci at $(\pm c, 0)$ and a positive number a, then we can use the definition of a hyperbola to derive the following equation of a hyperbola in which the constant difference between the distances to the foci is $2a$.

Equation of a Hyperbola Centered at (0, 0)

A hyperbola centered at $(0, 0)$ with foci $(c, 0)$ and $(-c, 0)$ and constant difference $2a$ has equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where a, b, and c are positive real numbers such that $c^2 = a^2 + b^2$.

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$(0, b)$

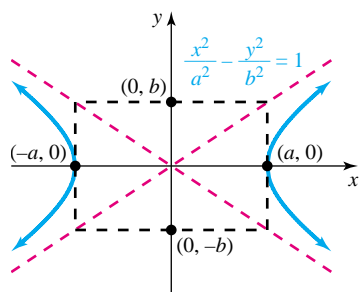$(-a, 0)$ $(a, 0)$

$(0, -b)$

FIGURE 12.28

The graph of a general equation for a hyperbola is shown in Fig. 12.28. Notice that the fundamental rectangle extends to the x-intercepts along the x-axis and

extends *b* units above and below the origin along the *y*-axis. The facts necessary for graphing a hyperbola centered at the origin and opening to the left and to the right are listed as follows.

Graphing a Hyperbola Centered at the Origin, Opening Left and Right

To graph the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$:

1. Locate the *x*-intercepts at $(a, 0)$ and $(-a, 0)$.
2. Draw the fundamental rectangle through $(\pm a, 0)$ and $(0, \pm b)$.
3. Draw the extended diagonals of the rectangle to use as asymptotes.
4. Draw the hyperbola to the left and right approaching the asymptotes.

E X A M P L E   3

A hyperbola opening left and right

Sketch the graph of $\frac{x^2}{36} - \frac{y^2}{9} = 1$, and find the equations of its asymptotes.

Solution

The *x*-intercepts are $(6, 0)$ and $(-6, 0)$. Draw the fundamental rectangle through these *x*-intercepts and the points $(0, 3)$ and $(0, -3)$. Extend the diagonals of the fundamental rectangle to get the asymptotes. Now draw a hyperbola passing through the *x*-intercepts and approaching the asymptotes as shown in Fig. 12.29. From the graph in Fig. 12.29 we see that the slopes of the asymptotes are $\frac{1}{2}$ and $-\frac{1}{2}$. Because the *y*-intercept for both asymptotes is the origin, their equations are $y = \frac{1}{2}x$ and $y = -\frac{1}{2}x$.

c a l c u l a t o r

c l o s e - u p

To graph the hyperbola and its asymptotes from Example 3, graph
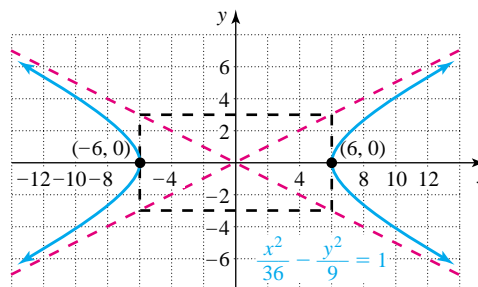
$y_1 = \sqrt{x^2/4 - 9}$, $y_2 = -y_1$, $y_3 = 0.5x$, and $y_4 = -y_3$.

F I G U R E   12.29

A hyperbola may open up and down. In this case the graph intersects only the *y*-axis. The facts necessary for graphing a hyperbola that opens up and down are summarized on the next page.

h e l p f u l   h i n t

We could include here general formulas for the equations of the asymptotes, but that is not necessary. It is easier first to draw the asymptotes as suggested and then to figure out their equations by looking at the graph.

Graphing a Hyperbola Centered at the Origin, Opening Up and Down

To graph the hyperbola $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$:

1. Locate the y-intercepts at $(0, b)$ and $(0, -b)$.
2. Draw the fundamental rectangle through $(0, \pm b)$ and $(\pm a, 0)$.
3. Draw the extended diagonals of the rectangle to use as asymptotes.
4. Draw the hyperbola opening up and down approaching the asymptotes.

E X A M P L E   4

A hyperbola opening up and down

Graph the hyperbola $\frac{y^2}{9} - \frac{x^2}{4} = 1$ and find the equations of its asymptotes.

Solution

If $y = 0$, we get

$$-\frac{x^2}{4} = 1$$
$$x^2 = -4.$$

Because this equation has no real solution, the graph has no x-intercepts. Let $x = 0$ to find the y-intercepts:

$$\frac{y^2}{9} = 1$$
$$y^2 = 9$$
$$y = \pm 3$$

F I G U R E   1 2 . 3 0

The y-intercepts are $(0, 3)$ and $(0, -3)$, and the hyperbola opens up and down. From $a^2 = 4$ we get $a = 2$. So the fundamental rectangle extends to the intercepts $(0, 3)$ and $(0, -3)$ on the y-axis and to the points $(2, 0)$ and $(-2, 0)$ along the x-axis. We extend the diagonals of the rectangle and draw the graph of the hyperbola as shown in Fig. 12.30. From the graph in Fig. 12.30 we see that the asymptotes have slopes $\frac{3}{2}$ and $-\frac{3}{2}$. Because the y-intercept for both asymptotes is the origin, their equations are $y = \frac{3}{2}x$ and $y = -\frac{3}{2}x$. ■
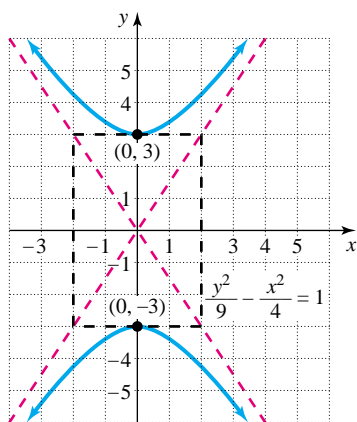
E X A M P L E   5

A hyperbola not in standard form

Sketch the graph of the hyperbola $4x^2 - y^2 = 4$.

Solution

First write the equation in standard form. Divide each side by 4 to get

$$x^2 - \frac{y^2}{4} = 1.$$

There are no y-intercepts. If $y = 0$, then $x = \pm 1$. The hyperbola opens left and right with x-intercepts at $(1, 0)$ and $(-1, 0)$. The fundamental rectangle extends to the intercepts along the x-axis and to the points $(0, 2)$ and $(0, -2)$ along the y-axis. We extend the diagonals of the rectangle for the asymptotes and draw the graph as shown in Fig. 12.31. ■
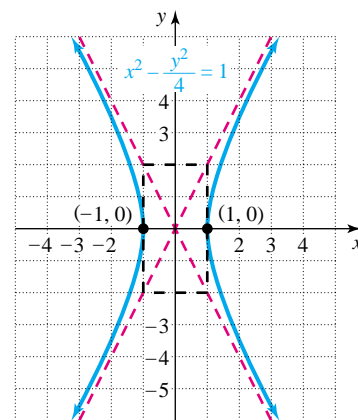

F I G U R E   1 2 . 3 1

WARM-UPS

True or false? Explain your answer.

1. The x-intercepts of the ellipse $\frac{x^2}{36} + \frac{y^2}{25} = 1$ are $(5, 0)$ and $(-5, 0)$.

2. The graph of $\frac{x^2}{9} + \frac{y}{4} = 1$ is an ellipse.

3. If the foci of an ellipse coincide, then the ellipse is a circle.

4. The graph of $2x^2 + y^2 = 2$ is an ellipse centered at the origin.

5. The y-intercepts of $x^2 + \frac{y^2}{3} = 1$ are $\left(0, \sqrt{3}\right)$ and $\left(0, -\sqrt{3}\right)$.

6. The graph of $\frac{x^2}{9} + \frac{y}{4} = 1$ is a hyperbola.

7. The graph of $\frac{x^2}{25} - \frac{y^2}{16} = 1$ has y-intercepts at $(0, 4)$ and $(0, -4)$.

8. The hyperbola $\frac{y^2}{9} - x^2 = 1$ opens up and down.

9. The graph of $4x^2 - y^2 = 4$ is a hyperbola.

10. The asymptotes of a hyperbola are the extended diagonals of a rectangle.

12.4   EXERCISES

Reading and Writing *After reading this section, write out the answers to these questions. Use complete sentences.*

1. What is the definition of an ellipse?

2. How can you draw an ellipse with a pencil and string?

3. Where is the center of an ellipse?

4. What is the equation of an ellipse centered at the origin?

5. What is the equation of an ellipse centered at (h, k)?

6. What is the definition of a hyperbola?

7. How do you find the asymptotes of a hyperbola?

8. What is the equation of a hyperbola centered at the origin and opening left and right?

Sketch the graph of each ellipse. See Example 1.

9. $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

10. $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$

11. $\dfrac{x^2}{9} + y^2 = 1$

12. $x^2 + \dfrac{y^2}{4} = 1$

13. $\dfrac{x^2}{36} + \dfrac{y^2}{25} = 1$

14. $\dfrac{x^2}{25} + \dfrac{y^2}{49} = 1$

Sketch the graph of each ellipse. See Example 2.

23. $\dfrac{(x-3)^2}{4} + \dfrac{(y-1)^2}{9} = 1$ **24.** $\dfrac{(x+5)^2}{49} + \dfrac{(y-2)^2}{25} = 1$

15. $\dfrac{x^2}{24} + \dfrac{y^2}{5} = 1$

16. $\dfrac{x^2}{6} + \dfrac{y^2}{17} = 1$

25. $\dfrac{(x+1)^2}{16} + \dfrac{(y-2)^2}{25} = 1$ **26.** $\dfrac{(x-3)^2}{36} + \dfrac{(y+4)^2}{64} = 1$

17. $9x^2 + 16y^2 = 144$ **18.** $9x^2 + 25y^2 = 225$

27. $(x-2)^2 + \dfrac{(y+1)^2}{36} = 1$ **28.** $\dfrac{(x+3)^2}{9} + (y+1)^2 = 1$

19. $25x^2 + y^2 = 25$ **20.** $x^2 + 16y^2 = 16$

Sketch the graph of each hyperbola and write the equations of its asymptotes. See Examples 3–5.

29. $\dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$

30. $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$

21. $4x^2 + 9y^2 = 1$ **22.** $25x^2 + 16y^2 = 1$

31. $\dfrac{y^2}{4} - \dfrac{x^2}{25} = 1$

32. $\dfrac{y^2}{9} - \dfrac{x^2}{16} = 1$

39. $x^2 - y^2 = 1$

40. $y^2 - x^2 = 1$

33. $\dfrac{x^2}{25} - y^2 = 1$

34. $x^2 - \dfrac{y^2}{9} = 1$

Graph both equations of each system on the same coordinate axes. Use elimination of variables to find all points of intersection.

41. $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

$x^2 - \dfrac{y^2}{9} = 1$

35. $x^2 - \dfrac{y^2}{25} = 1$

36. $\dfrac{x^2}{9} - y^2 = 1$

42. $x^2 - \dfrac{y^2}{4} = 1$

$\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

37. $9x^2 - 16y^2 = 144$

38. $9x^2 - 25y^2 = 225$

43. $\dfrac{x^2}{4} + \dfrac{y^2}{16} = 1$

$x^2 + y^2 = 1$

44. $x^2 + \dfrac{y^2}{9} = 1$
  $x^2 + y^2 = 4$

45. $x^2 + y^2 = 4$
  $x^2 - y^2 = 1$

46. $x^2 + y^2 = 16$
  $x^2 - y^2 = 4$

47. $x^2 + 9y^2 = 9$
  $x^2 + y^2 = 4$

48. $x^2 + y^2 = 25$
  $x^2 + 25y^2 = 25$

49. $x^2 + 9y^2 = 9$
  $y = x^2 - 1$

50. $4x^2 + y^2 = 4$
  $y = 2x^2 - 2$

51. $9x^2 - 4y^2 = 36$
  $2y = x - 2$

52. $25y^2 - 9x^2 = 225$
  $y = 3x + 3$

Solve each problem.

53. *Marine navigation.* The loran (long-range navigation) system is used by boaters to determine their location at sea. The loran unit on a boat measures the difference in time that it takes for radio signals from pairs of fixed points to reach the boat. The unit then finds the equations of two hyperbolas that pass through the location of the boat. Suppose a boat is located in the first quadrant at the intersection of $x^2 - 3y^2 = 1$ and $4y^2 - x^2 = 1$.

a) Use the graph on the next page to approximate the location of the boat.

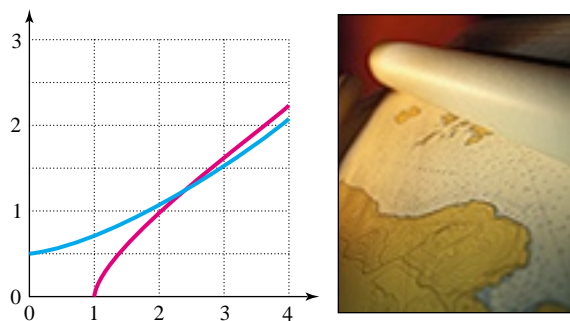**b)** Algebraically find the exact location of the boat.

FIGURE FOR EXERCISE 53

54. *Sonic boom.* An aircraft traveling at supersonic speed creates a cone-shaped wave that intersects the ground along a hyperbola, as shown in the accompanying figure. A thunderlike sound is heard at any point on the hyperbola. This sonic boom travels along the ground, following the aircraft. The area where the sonic boom is most noticeable is called the *boom carpet.* The width of the boom carpet is roughly five times the altitude of the aircraft. Suppose the equation of the hyperbola in the figure is

$$\frac{x^2}{400} - \frac{y^2}{100} = 1,$$

where the units are miles and the width of the boom carpet is measured 40 miles behind the aircraft. Find the altitude of the aircraft.
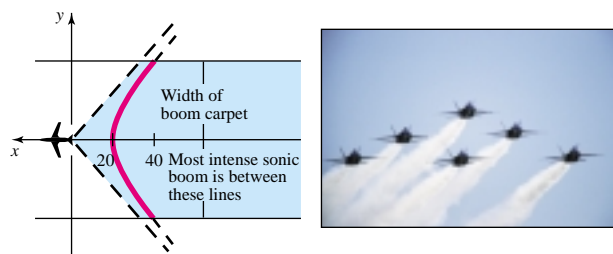

FIGURE FOR EXERCISE 54

GETTING MORE INVOLVED

55. *Cooperative learning.* Let (x, y) be an arbitrary point on an ellipse with foci $(c, 0)$ and $(-c, 0)$ for $c > 0$. The following equation expresses the fact that the distance from (x, y) to $(c, 0)$ plus the distance from (x, y) to $(-c, 0)$ is the constant value $2a$ (for $a > 0$):

$$\sqrt{(x - c)^2 + (y - 0)^2} + \sqrt{(x - (-c))^2 + (y - 0)^2} = 2a$$

Working in groups, simplify this equation. First get the radicals on opposite sides of the equation, then square both sides twice to eliminate the square roots. Finally, let $b^2 = a^2 - c^2$ to get the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

56. *Cooperative learning.* Let (x, y) be an arbitrary point on a hyperbola with foci $(c, 0)$ and $(-c, 0)$ for $c > 0$. The following equation expresses the fact that the distance from (x, y) to $(c, 0)$ minus the distance from (x, y) to $(-c, 0)$ is the constant value $2a$ (for $a > 0$):

$$\sqrt{(x - c)^2 + (y - 0)^2} - \sqrt{(x - (-c))^2 + (y - 0)^2} = 2a$$

Working in groups, simplify the equation. You will need to square both sides twice to eliminate the square roots. Finally, let $b^2 = c^2 - a^2$ to get the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

GRAPHING CALCULATOR EXERCISES

57. Graph $y_1 = \sqrt{x^2 - 1}$, $y_2 = -\sqrt{x^2 - 1}$, $y_3 = x$, and $y_4 = -x$ to get the graph of the hyperbola $x^2 - y^2 = 1$ along with its asymptotes. Use the viewing window $-3 \le x \le 3$ and $-3 \le y \le 3$. Notice how the branches of the hyperbola approach the asymptotes.

58. Graph the same four functions in Exercise 57, but use $-30 \le x \le 30$ and $-30 \le y \le 30$ as the viewing window. What happened to the hyperbola?

12.5 SECOND-DEGREE INEQUALITIES

In this section we graph second-degree inequalities and systems of inequalities involving second-degree inequalities.

Graphing a Second-Degree Inequality

In this section

- Graphing a Second-Degree Inequality
- Systems of Inequalities

A second-degree inequality is an inequality involving squares of at least one of the variables. Changing the equal sign to an inequality symbol for any of the equations of the conic sections gives us a second-degree inequality. Second-degree inequalities are graphed in the same manner as linear inequalities.