The Trapezoidal Rule

This technique is a much more accurate way to approximate area beneath a curve. To construct the trapezoids, you mark the height of the function at the beginning and end of the width interval, then connect the two points. However, this method requires you to memorize a formula.

DEFINITION

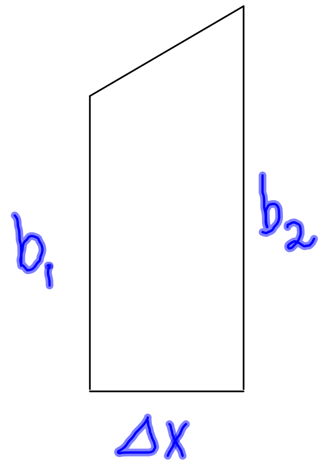
Let *f* be continuous on [*a*, *b*].
The Trapezoidal Rule for approximating $\int_a^b f(x)\,dx$ is given by

$$\frac{b-a}{2n}\left[f(a)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(b)\right]$$

The area of any trapezoid is one half of the height times the sum of the bases (the bases are the parallel sides.) Recall the area formula A =h/2(b_1 + b_2). The reason you see all those 2's in the Trapezoidal Rule is that every base is used twice for consecutive trapezoids except for the bases at the endpoints.

CAUTION:
The Trapezoidal Rule's Formula contains the expression $\frac{b-a}{2n}$

You still use the formula $\frac{b-a}{n}$ to find the width of the trapezoids.



The coefficients in the Trapezoidal Rule follow the pattern:

1 2 2 2 . . . 2 2 1

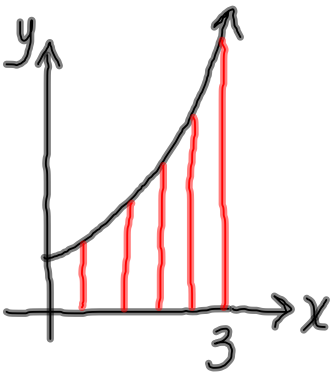
The Trapezoidal Rule A Second Glimpse:

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right]$$

where [*a*, *b*] is partitioned into *n* subintervals of equal length. $h = \frac{b-a}{n}$

NOTE: The Trapezoidal Rule overestimates a curve that is concave up and underestimates functions that are concave down.

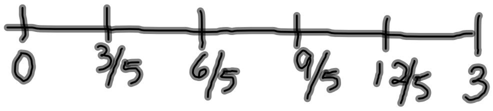
EX #1:
Approximate the area beneath $f(x) = x^2 + 1$ on the interval [0, 3] using the Trapezoidal Rule with n = 5 trapezoids.



1) Sketch:

2) $\Delta x = \frac{b-a}{n} = \frac{3-0}{5} = \frac{3}{5}$

3) Intervals:



$$T = \frac{b-a}{2n}\left[y_0 + 2y_1 + 2y_2 + 2y_3 + 2y_4 + y_5\right]$$

$$T = \frac{3}{5(2)}\left[1 + 2\left(\frac{34}{25}\right) + 2\left(\frac{61}{25}\right) + 2\left(\frac{106}{25}\right) + 2\left(\frac{169}{25}\right) + 10\right]$$

$$T = \frac{3}{10}\left[1 + \frac{68}{25} + \frac{122}{25} + \frac{212}{25} + \frac{338}{25} + 10\right]$$

$$T = \frac{3}{10}\left[\frac{25 + 68 + 122 + 212 + 338 + 250}{25}\right]$$

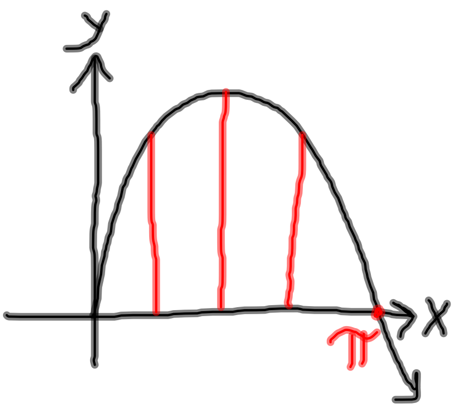
$$T = \frac{3}{10}\left[\frac{1015}{25}\right]$$

$$T = \frac{609}{50} = 12.18$$

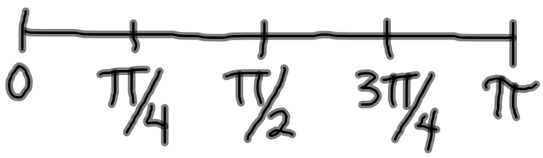
The approximate area between the curve and the x-axis is the sum of the four trapezoids.

This is a trapezoidal approximation, not a Reimann sum approximation. Reimann sum refers only to an approximation with rectangles.

EX #2: Approximate the area beneath y = sin x on the interval [o, π] using the Trapezoidal Rule with n = 4 trapezoids.



$$\Delta x = \frac{\pi - 0}{4} = \frac{\pi}{4}$$



$$T = \frac{b-a}{2n}\left[y_0 + 2y_1 + 2y_2 + 2y_3 + y_4\right]$$

$$T = \frac{\pi}{4(2)}\left[0 + 2\left(\frac{\sqrt{2}}{2}\right) + 2(1) + 2\left(\frac{\sqrt{2}}{2}\right) + 0\right]$$

$$T = \frac{\pi}{8}\left[0 + \sqrt{2} + 2 + \sqrt{2} + 0\right]$$

$$T = \frac{\pi}{8}\left[2 + 2\sqrt{2}\right]$$

$$\underline{\underline{T \approx 1.896}}$$

Check with calculator, in radian mode:

$$\text{fnInt}(\sin x, x, 0, \pi) = 2$$

Simpson's Rule

Another area-approximating tool is Simpson's Rule. Geometrically, it creates tiny parabolas to wrap closer around the function we're approximating. The formula is similar to the Trapezoidal Rule, with a small catch...
you can only use an even number of subintervals.

DEFINITION

Let *f* be continuous on [*a*, *b*].
Simpson's Rule for approximating $\int_a^b f(x)\,dx$ is given by

$$\frac{b-a}{3n}\left[f(a)+4f(x_1)+2f(x_2)+\cdots+4f(x_{n-1})+f(b)\right]$$

The coefficients in Simpson's Rule follow the pattern:

1 4 2 4 2 . . . 4 2 4 1

Simpson's Rule A Second Glimpse:

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[y_0 + 4y_1 + 2y_2 + 4y_3 \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right]$$

where [*a*, *b*] is partitioned into *n* even subintervals of equal length. $h = \dfrac{b-a}{n}$

EX #3:
Approximate the area beneath $f(x) = x^2 + 1$ on the interval [0, 3] using Simpson's Rule with *n* = 6 subintervals.

$S = \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6]$

$h = \frac{b-a}{n}$

$h = \frac{3-0}{6} = \frac{1}{2}$

x	0	1/2	1	3/2	2	5/2	3
$y = x^2+1$	1	$\frac{5}{4}$	2	$\frac{13}{4}$	5	$\frac{29}{4}$	10

$S = \frac{1}{2(3)}\left[1 + 4\left(\frac{5}{4}\right) + 2(2) + 4\left(\frac{13}{4}\right) + 2(5) + 4\left(\frac{29}{4}\right) + 10\right]$

$S = \frac{1}{6}[1 + 5 + 4 + 13 + 10 + 29 + 10]$

$S = \frac{1}{6}[72]$

$S = 12$

fnInt $(x^2+1, x, 0, 3) = 12$

EX #4: Approximate the area beneath y = sin x on the interval [o, π] using the Trapezoidal Rule with n = 6 subintervals.

$$S = \frac{h}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6\right]$$

$$h = \frac{b-a}{n}$$

$$h = \frac{\pi - 0}{6} = \frac{\pi}{6}$$

x	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π
$y = \sin x$	0	$1/2$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0

$$S = \frac{\pi}{6(3)}\left[0 + 4\left(\tfrac{1}{2}\right) + 2\left(\tfrac{\sqrt{3}}{2}\right) + 4(1) + 2\left(\tfrac{\sqrt{3}}{2}\right) + 4\left(\tfrac{1}{2}\right) + 0\right]$$

$$S = \frac{\pi}{18}\left[0 + 2 + \sqrt{3} + 4 + \sqrt{3} + 2 + 0\right]$$

$$S = \frac{\pi}{18}\left[8 + 2\sqrt{3}\right]$$

$$S = \frac{\pi(4+\sqrt{3})}{9}$$

$$S \approx 2.001$$

fnInt(sin x, x, 0, 3) ≈ 1.9899

Summary:

Riemann sums use rectangles to approximate the area beneath a curve. The heights of the rectangles are based on the height of the function at the left end, right end, or midpoint of each subinterval.

The width of each subinterval in all the approximating techniques is

$$\Delta x = \frac{b-a}{n}$$

The Trapezoidal Rule is the average of the left and right sums, and usually gives a better approximation than either does individually.

Simpson's Rule uses intervals topped with parabolas to approximate area; therefore, it gives the exact area beneath quadratic functions.