Lecture # 44

Alternating Series; Conditional convergence

- So far we have seen infinite series that have positive terms only.
- We have also defined the Limit and the Sum of such infinite series
- Now we look at series that have terms which have alternating signs, known as **Alternating series**

Example

1) 1-1+1-1+…
2) 1+2-3+4….

More generally, an alternating series has one of the two following forms

$$\sum_{k=1}^{\infty}(-1)^{k+1}a_k = a_1 - a_2 + a_3 - a_4 + \ldots \qquad (1)$$

$$\sum_{k=1}^{\infty}(-1)^{k}a_k = -a_1 + a_2 - a_3 + a_4 - \ldots \qquad (2)$$

Note that all the terms a_k are to be taken as being positive.
The following theorem is the key result on convergence of alternating series.

Theorem11.7.1
(Alternating Series Test)
An alternating series of either form (1) or (2) converges if the following two conditions are satisfied:

(a) $a_1 > a_2 > a_3 \ldots\ldots > a_k > \ldots\ldots$
(b) $\lim_{k\to+\infty} a_k = 0$

This theorem tells us when an alternating series converges.

Example

Use the alternating series test to show that the following series converge

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}$$

Solution:

The two conditions in the alternating series test are satisfied since $a_k = \frac{1}{k} > \frac{1}{k+1} = a_{k+1}$

and $\lim_{k\to+\infty} a_k = \lim_{k\to+\infty}\frac{1}{k} = 0$

Note that this series is the Harmonic series, but alternates. It's called the Alternating Harmonic Series.
The harmonic series diverges. The alternating Harmonic series converges.
Now we will look at the errors involved in approximating an alternating series with a partial sum.

Theorem 11.7.2
If an alternating series satisfies the conditions of the alternating series test, and if the sum S of the series is approximated by the nth partial sum S_n , thereby resulting in an error of $S- S_n$,then

$$| S- S_n | < a_{n+1}$$

Moreover, the sign of error is the same as that of the coefficient of a_{n+1}in the series.

Example
The alternating series

$$1 – 1/2 + 1/3 – 1/4 + ……. + (-1)^{k+1} 1/k + …….$$

satisfies the condition of the alternating series test ; hence the series has a sum S, which we know must lie between any two successive partial sums. In particular, it must lie between

$$S_7 = 1 – 1/2 + 1/3 – 1/4 + 1/5 – 1/6 + 1/7 = 319/420$$

And

$$S_7 = 1 – 1/2 + 1/3 – 1/4 + 1/5 – 1/6 + 1/7 – 1/8 = 533/840$$

So

$$533/840 < S < 319/420$$

If we take S = ln2 then

$$533/840 < \ln 2 < 319/420$$
$$0.6345 < \ln 2 < 0.7596$$

The value of ln2, rounded to four decimal places, is .6931, which is consistent with these inequalities. It follows from Theorem 11.7.2 that

$$| \ln 2 – S_7 | = | \ln 2 – 319/420 | < a_8 =1/8$$

And

$$| \ln 2 – S_8 | = | \ln 2 – 533/840 | < a_9 =1/9$$

Absolute and Conditional Convergence

The series

$$1 – 1/2 - 1/2^2 +1/2^3 + 1/2^4 – 1/2^5 – 1/2^6 + ………….$$

does not fit in any of the categories studied so far, it has mixed signs , but is not alternating. We shall now develop some convergence tests that can be applied to such series.

Definition 11.7.3

A series $\sum_{k=1}^{\infty} u_k = u_1 + u_2 + \dots\dots\dots\dots + u_k + \dots\dots$

is said to converge absolutely, if the series of absolutes values

$\sum_{k=1}^{\infty} |u_k| = |u_1| + |u_2| + \dots\dots\dots\dots + |u_k| + \dots\dots$ converges.

Example

The series

$1 - 1/2 - 1/2^2 + 1/2^3 + 1/2^4 - 1/2^5 - 1/2^6 + \dots\dots\dots\dots$

Converges absolutely since the series of absolute values

$1 + 1/2 + 1/2^2 + 1/2^3 + 1/2^4 + 1/2^5 + 1/2^6 + \dots\dots\dots$

is a convergent geometric series.

On the other hand, the alternating harmonic series

1 - 1/2 + 1/3 - 1/4 + 1/5 - …………

does not converge absolutely since the series of absolute values

1 + 1/2 + 1/3 + 1/4 + 1/5 + …………

diverges.

Absolute convergence is of importance because of the following theorem.

Theorem 11.7.4

If the series $\sum_{k=1}^{\infty} |u_k| = |u_1| + |u_2| + \dots\dots\dots\dots + |u_k| + \dots\dots$

Converges, then so does the series $\sum_{k=1}^{\infty} u_k = u_1 + u_2 + \dots\dots\dots\dots + u_k + \dots\dots$

In other words, if a series converges absolutely, then it converges.

Since the series

$1 - 1/2 - 1/2^2 + 1/2^3 + 1/2^4 - 1/2^5 - 1/2^6 + \dots\dots\dots\dots$

Converges absolutely. It follows from Theorem 11.7.4 that the given series converges.

Example

Show that the series $\sum_{k=1}^{\infty} \frac{\cos k}{k^2}$ converges.

Solution: Since $|\cos k| \le 1$ for all k,

$$\left|\frac{\cos k}{k^2}\right| \le \frac{1}{k^2}$$

thus

$$\left|\sum_{k=1}^{\infty}\frac{\cos k}{k^2}\right| \sum_{k=1}^{\infty}\frac{\cos k}{k^2}$$

converges by the comparison test, and consequently

$$\sum_{k=1}^{\infty}\frac{\cos k}{k^2} \quad \text{converges.}$$

If $\sum |u_k|$ diverges, no conclusion can be drawn about the convergence or divergence of $\sum u_k$

For example, consider the two series

$1 - 1/2 + 1/3 - 1/4 + \ldots\ldots + (-1)^{k+1}\, 1/k + \ldots\ldots$ (A)

$- 1 - 1/2 - 1/3 - 1/4 - \ldots\ldots - 1/k - \ldots\ldots$ (B)

Series (A), the alternating harmonic series, converges, whereas series (B), being a constant times the harmonic series, diverges.

Yet in each case the series of absolute values is

$1 + 1/2 + 1/3 + \ldots\ldots + 1/k + \ldots\ldots$

which diverges. A series such as (A), which is convergent, but not absolutely convergent, is called **conditionally convergent**.

Theorem 11.7.5

(Ratio Test for Absolute Convergence)

Let $\sum u_k$ be a series with nonzero terms and suppose that $\rho = \lim_{k\to\infty}\frac{|u_{k+1}|}{|u_k|}$

(a) If $\rho < 1$, the series $\sum u_k$ converges absolutely and therefore converges.

(b) If $\rho > 1$ or $\rho = +\infty$, then the series $\sum u_k$ diverges.

(c) If $\rho = 1$, no conclusion about convergence or absolute convergence can be drawn from this test.

EXAMPLE

The series

$$\sum_{k=1}^{\infty}(-1)^k\frac{2^k}{k!}$$

converges absolutely since

$$\rho = \lim_{k\to+\infty}\frac{|u_{k+1}|}{|u_k|} = \lim_{k\to+\infty}\frac{2^{k+1}}{(k+1)!}\cdot\frac{k!}{2^k}$$

$$= \lim_{k\to+\infty}\frac{2}{k+1} = 0 < 1$$

Power Series in x

If c_0, c_1, c_2, ………. are constants and x is a variable, then a series of the form

$$\sum_{k=o}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + ... + c_k x^k + ...$$ is called a power series in x.

Some examples of power series in x are

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + ...$$

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

$$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$$

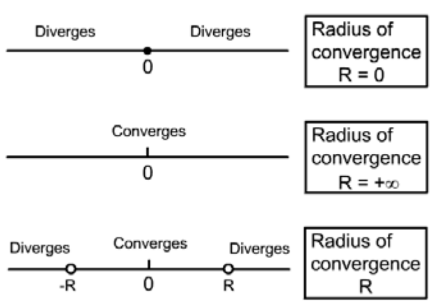
Theorem 11.8.1

For any power series in x, exactly one of the following is true:

(a) The series converges only for x=0
(b) The series converges absolutely (and hence converges) for all real values of x.
(c) The series converges absolutely (and hence converges) for all x in some finite open interval (-R, R), and diverges if x<-R or x>R. At either of the points x = R or x = -R, the series may converge absolutely, converge conditionally, or diverge, depending on the particular series.

Radius and Interval of Convergence

Theorem 11.8.1 states that the set of values for which a power series in x converges is always an interval centered at 0; we call this the interval of convergence, corresponding to this interval series has radius called radius of convergence.



Example:

Find the interval of convergence and radius of convergence of the following power series.

$$\sum_{k=1}^{\infty} x^k$$

Solution: We shall apply the ratio test for absolute convergence. We have

$$\rho = \lim_{k\to\infty}\left|\frac{u_{k+1}}{u_k}\right| = \lim_{k\to\infty}\left|\frac{x^{k+1}}{x^k}\right| = \lim_{k\to\infty}|x| = |x|$$

So the ratio test for absolute convergence implies that the series converges absolutely if $\rho = |x| < 1$ and diverges if $\rho = |x| > 1$ .The test is inconclusive if | x | =1 (i.e. x = 1 or x = -1), so convergence at these points must be investigated separately . At these points the series becomes

$$\sum_{k=0}^{\infty} 1^k = 1+1+1+1..... \qquad x=1$$

$$\sum_{k=0}^{\infty} (-1)^k = 1-1+1-1+...... \qquad x=-1$$

Both of which diverge; thus, the interval of convergence for the given power series is (-1,1), and the radius of convergence is R = 1.

Power series in x-a

$$\sum_{k=0}^{\infty} c_k(x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + ... + c_k(x-a)^k + ...$$

This series is called power series in x-a. Some examples are

$$\sum_{k=0}^{\infty} \frac{(x-1)^k}{k+1} = 1 + \frac{(x-1)}{2} + \frac{(x-1)^2}{3} + \frac{(x-1)^3}{4} + ... \qquad a=1$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k (x+3)}{k!} = 1-(x+3) + \frac{(x+3)^2}{2!} - \frac{(x+3)^3}{3!} + ... \qquad a=-3$$

Theorem 11.8.1

For any power series in $\sum c_k(x-a)^k$, exactly one of the following is true:

(a) The series converges only for x=a
(b) The series converges absolutely (and hence converges) for all real values of x.
(c) The series converges absolutely (and hence converges) for all x in some finite open interval (a-R,a+R), and diverges if x<a-R or x>a+R. At either of the points x =a-R or x =a+R, the series may converge absolutely, converge conditionally, or diverge, depending on the particular series.

It follows from this theorem that now interval of convergence is centered at x=a.