Lecture # 41

Sequences and Monotone Sequences

- Definition of a sequence
- Graphs of sequences
- Limit of a sequence
- Recursive sequence
- Testing for monotonicity
- Eventually monotonic sequences
- Convergence of monotonic sequence
- More on convergence intuitively

Definition of a sequence

A sequence in math is a succession of numbers e.g 2,4,6,8,……. & 1,2,3,4,…….

The numbers in a sequence are called terms of a sequence
Each term has a positional name, like 1st term, 2nd term etc and we can write them as a_1, a_2 etc.
It is convenient to write a sequence as a formula
2, 4, 6, 8,… can be written as a formula as $\{2n\}_{n=1}^{+\infty}$

EXAMPLE

List the first five terms of the sequence

$$\{2^n\}_{n=1}^{+\infty}$$

Substituting n = 1,2,3,4,5 into the formula we get

$$2^1, 2^2, 2^3, 2^4, 2^5$$

or equivalently,

$$2,4,8,16,32$$

We will write a sequences like a_1 , a_2,a_3….. as $\{a_n\}_{n=1}^{+\infty}$

Here is a formal definition of a sequence

DEFINITION 11.1.1

A sequence or infinite sequence is a function whose domain is the set of positive integers; that is ,

$$\{a_n\}_{n=1}^{+\infty}$$

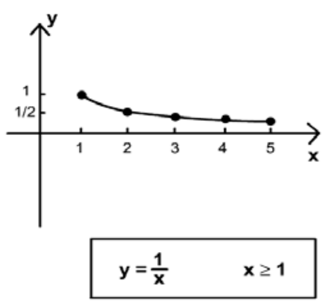
which is an alternative notation for the function

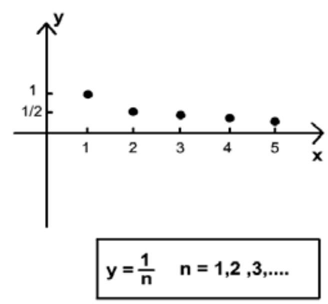
$f(n) = a_n$ where $n = 1,2,3$ ....

Graphs of Sequences

As we see that sequences are functions, we can talk about the graphs of them.

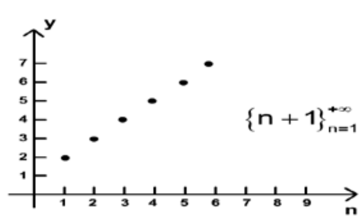
The graph of the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{+\infty}$ is the graph of the function equation y = 1/n for n = 1, 2,3….

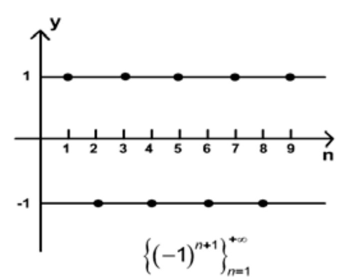


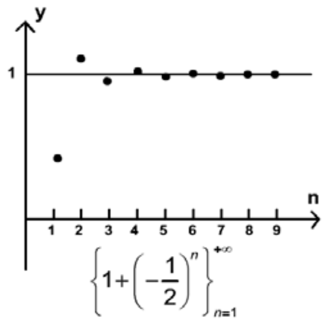


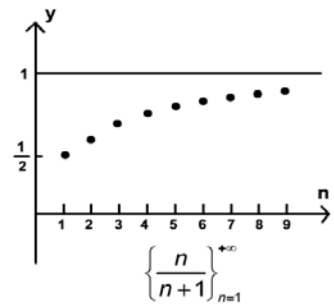
Limit of a sequence

Here are graphs of four sequences, each of which behaves differently as n increases



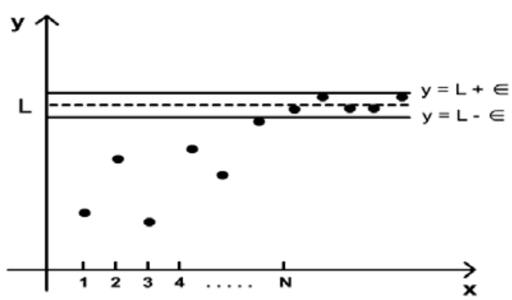






Some of these graphs have the concept of a limit in them.

A sequence an converges to a limit L if for any positive number E there is a point in the sequences after which all terms lie btw lines L+E and L-E



From point N on the terms in the sequence are all within ∈ units of L.

DEFINITION 11.1.2

A sequence $\{a_n\}$ is said to *converge* to the limit L if given any $\in > 0$, there is poitive integer N such that $|a_n - L| < \in$ for $n \geq N$. In this case we write

$$\lim_{n\to+\infty} a_n = L$$

A sequence that does not converge to some finite limit is said to *diverge.*

As we apply limit the following two sequences converge

$$\lim_{n\to+\infty} \frac{n}{(n+1)} = 1$$

and

$$\lim_{n\to+\infty} \left(1+\left(-\frac{1}{2}\right)^n\right) = 1$$

The following theorem shows that the familiar properties of limits apply to sequence

THEOREM 11.1.3

Suppose that the sequences $\{a_n\}$ and $\{b_n\}$ converge to limits L_1 and L_2 respectively, and c is a contant . Then

a) $\lim_{n\to+\infty} c = c$

b) $\lim_{n\to+\infty} ca_n = c \lim_{n\to+\infty} a_n = cL_1$

c) $\lim_{n\to+\infty} (a_n + b_n) = \lim_{n\to+\infty} a_n + \lim_{n\to+\infty} b_n = L_1 + L_2$

d) $\lim_{n\to+\infty} (a_n - b_n) = \lim_{n\to+\infty} a_n - \lim_{n\to+\infty} b_n = L_1 - L_2$

e) $\lim_{n\to+\infty} (a_n b_n) = \lim_{n\to+\infty} a_n \lim_{n\to+\infty} b_n = L_1 L_2$

f) $\lim_{n\to+\infty} \left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to+\infty} a_n}{\lim_{n\to+\infty} b_n} = \frac{L_1}{L_2}$ (If $L_2 \neq 0$)

Determine whether the sequence converges or diverges

$$\left\{\frac{n}{2n+1}\right\}_{n=1}^{+\infty}$$

Dividing numerator and denomenator by n yields

$$\lim_{n\to+\infty}\frac{n}{2n+1}=\lim_{n\to+\infty}\frac{1}{2+1/n}$$

$$=\frac{\lim\limits_{n\to+\infty} 1}{\lim\limits_{n\to+\infty}(2+1/n)}$$

$$=\frac{\lim\limits_{n\to+\infty} 1}{\lim\limits_{n\to+\infty} 2+\lim\limits_{n\to+\infty} 1/n}$$

$$=\frac{1}{2+0}=\frac{1}{2}$$

Thus the sequence converges to 1/2

Recursive Sequence

Some sequences are defined by specifying one or more initial terms and giving a formula that relates each subsequent term to the term that precedes it , such sequences are said to be defined recursively .

EXAMPLE

Let $\{a_n\}$ be the sequence defined by $a_1 = 1$ and the recursion formula

$$a_{n+1}=\frac{1}{2}(a_n+3/a_n) \quad \text{for } n\geq 1$$

$$a_2=\frac{1}{2}(a_1+3/a_1)=\frac{1}{2}(1+3)=2 \qquad \boxed{n = 1}$$

$$a_3=\frac{1}{2}(a_2+3/a_2)=\frac{1}{2}(2+\frac{3}{2})=\frac{7}{4} \qquad \boxed{n = 2}$$

$$a_4=\frac{1}{2}(a_3+3/a_3)=\frac{1}{2}(\frac{7}{4}+\frac{12}{7})=\frac{97}{56} \qquad \boxed{n = 3}$$

Monotonicity and testing for monotonicity

DEFINITION 11.2.1

A sequence $\{a_n\}$ is called

Increasing if

$$a_1 < a_2 < a_3 < \ldots\ldots < a_n < \ldots.$$

Nondecreasing if

$$a_1 \le a_2 \le a_3 \le \ldots\ldots \le a_n \le \ldots.$$

Decreasing if

$$a_1 > a_2 > a_3 > \ldots\ldots > a_n > \ldots.$$

Nonincreasing if

$$a_1 \ge a_2 \ge a_3 \ge \ldots\ldots \ge a_n \ge \ldots.$$

A sequence that is either nondecreasing or nonincreasing is called monotone, and a sequence that is increasing or decreasing is called strictly monotone .Observe that a strictly monotone is monotone , but not conversely.

EXAMPLE

$\frac{1}{2},\frac{2}{3},\frac{3}{2},\ldots\ldots,\frac{n}{n+1},\ldots..$ is increasing

$1,\frac{1}{2},\frac{1}{3},\ldots..,\frac{1}{n},\ldots\ldots$ is decreasing

$1,1,2,2,3,3,\ldots\ldots$ is nondecreasing

$1,1,\frac{1}{2},\frac{1}{2},\frac{1}{3},\frac{1}{3},\ldots\ldots$ is nonincreasing

All four of these sequences are monotone, but the sequence

$$1,-\frac{1}{2},\frac{1}{3},-\frac{1}{4},\ldots\ldots,(-1)^{n+1}\frac{1}{n},\ldots\ldots$$

is not monotone. The first and second sequences are strictly monotone.

Testing for monotonicity

Monotone sequences are classified as follows

Difference between successive terms	Classification
$a_{n+1} - a_n > 0$	Increasing
$a_{n+1} - a_n < 0$	Decreasing
$a_{n+1} - a_n \geq 0$	Nondecreasing
$a_{n+1} - a_n \leq 0$	Nonincreasing

Monotone Sequence with positive terms are classified as follows

Ratio of successive terms	Classification
$a_{n+1} / a_n > 1$	Increasing
$a_{n+1} / a_n < 1$	Decreasing
$a_{n+1} / a_n \geq 1$	Nondecreasing
$a_{n+1} / a_n \leq 1$	Nonincreasing

If $f(n) = a_n$ is the nth term of a sequence, and if f is differentiable for x>=1 then we have the following results

Derivative of f for $x \geq 1$	Classification of the sequence with $f(x) = a_x$
$f'(x) > 0$	Increasing
$f'(x) < 0$	Decreasing
$f'(x) \geq 0$	Nondecreasing
$f'(x) \leq 0$	Nonincreasing

EXAMPLE

Show that $\frac{1}{2},\frac{2}{3},\frac{3}{4},\ldots\ldots,\frac{n}{n+1},\ldots\ldots.$

is an increasing sequence

It is intuitively clear that the sequence is increasing .To prove that this is so, let

$$a_n = \frac{n}{n+1}$$

We can obtain a_{n+1} by replacing n by n+1 in this formula. This yields

$$a_{n+1} = \frac{n+1}{(n+1)+1} = \frac{n+1}{n+2}$$

Thus for $n \geq 1$

$$a_{n+1} - a_n = \frac{n+1}{n+2} - \frac{n}{n+1}$$

$$= \frac{n^2+2n+1-n^2-2n}{(n+1)(n+2)}$$

$$= \frac{1}{(n+1)(n+2)} > 0$$

which proves that the sequence is increasing, so sequence is strictly monotone.

EXAMPLE

Show that the sequence

$$\frac{1}{2},\frac{2}{3},\frac{3}{4},\ldots\ldots,\frac{n}{n+1},\ldots\ldots.$$

is increasing by examining the ratio of successive terms.

As shown earlier

$$a_n = \frac{n}{n+1} \qquad \text{and} \qquad a_{n+1} = \frac{n+1}{n+2}$$

Thus

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)/(n+2)}{n/(n+1)}$$

$$= \frac{n+1}{n+2}.\frac{n+1}{n} = \frac{n^2+2n+1}{n^2+2n}$$

Since the numerator exceeds the denominator, the ratio exceeds 1, that is $a_{n+1}/a_n > 1$ for $n \geq 1$. This proves that the sequence is increasing.

Eventually monotonic sequences

A sequence an is eventually monotone if there is some integer N such that the sequence is monotone for *n* >= N

EXAMPLE

Show that the sequence $\left\{\frac{10^n}{n!}\right\}_{n=1}^{+\infty}$ is eventually decreasing.

We have

$$a_n = \frac{10^n}{n!} \quad \text{and} \quad a_{n+1} = \frac{10^{n+1}}{(n+1)!}$$

so

$$\frac{a_{n+1}}{a_n} = \frac{10^{n+1}/(n+1)!}{10^n/n!}$$

$$= \frac{10^{n+1}n!}{10^n(n+1)!} = 10\frac{n!}{(n+1)n!}$$

$$= \frac{10}{n+1}$$

This is clear that $a_{n+1}/a_n < 1$ for all $n \geq 10$, so the sequence is eventually decreasing.

Convergence of monotonic sequence:

THEOREM 11.2.2

If $a_1 \le a_2 \le a_3 \le ..... \le a_n \le .....$ is a nondecreasing sequence, then there are two possibilities:

a) There is a constant M, called a upper bound for the sequence, such that $a_n \le M$ for all n, in which case the sequence converges to limit L satisfying $L \le M$

b) No upper bound exists, in which case

$$\lim_{n \to +\infty} a_n = +\infty$$

THEOREM 11.2.3

If $a_1 \ge a_2 \ge a_3 \ge ...... \ge a_n \ge .. ..$ is a nonincreasing sequence, then there are two possibilities:

a) There is a constant M, called a lower bound for the sequence, such that $a_n \ge M$ for all n, in which case the sequence converges to limit L satisfying $L \ge M$

b) No lower bound exists, in which case

$$\lim_{n \to +\infty} a_n = -\infty$$

EXAMPLE

Show that the sequence $\left\{\frac{n}{n+1}\right\}_{n=1}^{+\infty}$ converges.

We showed earlier that the given sequence is increasing (hence nondecreasing).It is evident that the number M=1 is an upper bound for the sequence since

$$an = \frac{n}{n+1} < 1 \qquad n = 1,2,....$$

Thus, by Theorem 11.2.2 the sequence converges to some limit L such that $L \leq M = 1$.This is indeed the case since

$$\lim_{n \to +\infty} \frac{n}{n+1} = \lim_{n \to +\infty} \frac{1}{1+1/n} = 1$$

An Intuitive View of Convergence:

Informally stated, the convergence or divergence of a sequence does not depend on the behavior of the "initial terms" of the sequence, but rather on the behavior of the "tail end". Thus for sequence { an } to converge to a limit L, it does not matter if the initial terms are for from L, just so the terms in the sequence are eventually arbitrarily close to L. This being the case , one can add, delete, or alter finitely many terms without affecting the convergence, divergence, or the limit (if it exists).

