Lecture # 40

L'Hopital's Rule and Indeterminate forms

- L'Hopital's Rule and 0 / 0
- Indeterminate form of type ∞ / ∞
- Indeterminate form of type 0. ∞
- Indeterminate forms of type 0^0, ∞^0 and 1^∞ and ∞-∞

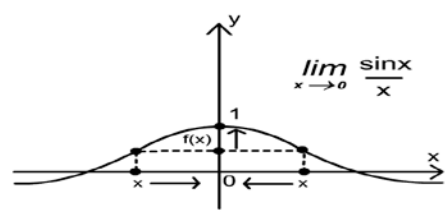
If we look at $\lim\limits_{x\to 2}\dfrac{x^2-4}{x-2}$ and $\lim\limits_{x\to 0}\dfrac{\sin x}{x}$

The top and the bottom both approach 0 and so these types of limits are called indeterminate forms of type 0/0.
This kind of limit can converge in which case it will have finite real values, or it can diverge.
The value, if it converges, is not obvious right away.
The first expression can be factored and bottom cancelled to get a finite value and the second expression requires geometrical observation as shown below

$$\lim_{x\to 2}\frac{x^2-4}{x-2}$$

$$=\lim_{x\to 2}\frac{(x-2)(x+2)}{(x-2)}$$

$$=\lim_{x\to 2}(x+2)=4$$



We need a more general concept that work almost always.
L'Hopital's Rule provide us solution that we are looking for

THEOREM 10.2.1
(L'Hopital's Rule for Form 0/0)

Let *lim* stand for one of the *limits*

$$\lim_{x\to a}\quad \lim_{x\to a^+}\quad \lim_{x\to a^-}\quad \lim_{x\to +\infty}\quad \lim_{x\to -\infty}$$

and suppose that *lim* f(x) = 0 and *lim* g(x) = 0.If *lim*[f′(x) / g′(x)] has a finite value L, or if this *limit* is +∞ or -∞ , then

$$\lim\frac{f(x)}{g(x)}=\lim\frac{f'(x)}{g'(x)}$$

How we can use this theorem, here are some steps to be followed

STEP1: Check that *lim* [f(x)/g(x)] is an intermediate form. If it is not , then L'Hopital's rule cannot be used.

STEP2: Differentiate f and g seperately.

STEP3: Find *lim* [f′(x)/g′(x)]. If this limit is finite, +∞ or -∞ , then it is equal to *lim* [f(x)/g(X)]

EXAMPLE

Use L'Hopital's rule to evaluate

a) $\lim\limits_{x\to 2}\dfrac{x^2-4}{x-2}$ b) $\lim\limits_{x\to 0}\dfrac{\sin 2x}{x}$

Since

$\lim\limits_{x\to 2}(x^2-4)=0$ and $\lim\limits_{x\to 2}(x-2)=0$

the given limit is an intermediate form of type 0/0. Thus,L'Hopital's rule applies and we can write

$$\lim_{x\to 2}\frac{(x^2-4)}{x-2}=\lim_{x\to 2}\frac{\frac{d}{dx}\left[x^2-4\right]}{\frac{d}{dx}\left[x-2\right]}$$

$$=\lim_{x\to 2}\frac{2x}{1}=4$$

b) $\lim\limits_{x\to 0}\dfrac{\sin 2x}{x}$

Since

$\lim\limits_{x\to 0}\sin 2x=0$ and $\lim\limits_{x\to 2}x=0$

the given limit is an intermediate form of type 0/0. Thus,L'Hopital's rule applies and we can write

$$\lim_{x\to 0}\frac{\sin 2x}{x}=\lim_{x\to 0}\frac{\frac{d}{dx}[\sin 2x]}{\frac{d}{dx}[x]}$$

$$=\lim_{x\to 0}\frac{2\cos 2x}{1}=2$$

EXAMPLE

Evaluate

$$\lim_{x\to \pi/2}\frac{1-\sin x}{\cos x}$$

$$\lim_{x\to \pi/2}(1-\sin x)=0 \qquad \lim_{x\to \pi/2}\cos x=0$$

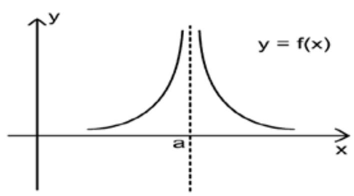
the given limit is an intermediate form of type 0/0.
Thus by , L'Hopital's rule

$$\lim_{x\to \pi/2}\frac{1-\sin x}{\cos x}=\lim_{x\to \pi/2}\frac{\frac{d}{dx}[1-\sin x]}{\frac{d}{dx}[\cos x]}$$

$$=\lim_{x\to \pi/2}\frac{-\cos x}{-\sin x}=\frac{0}{-1}=0$$

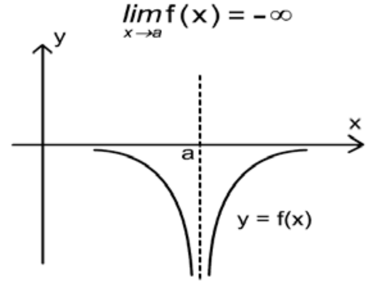
Some Notation

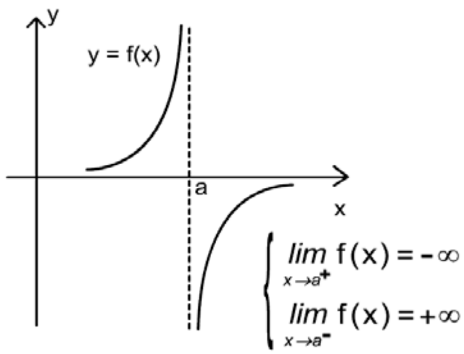
In the four parts of figure below,

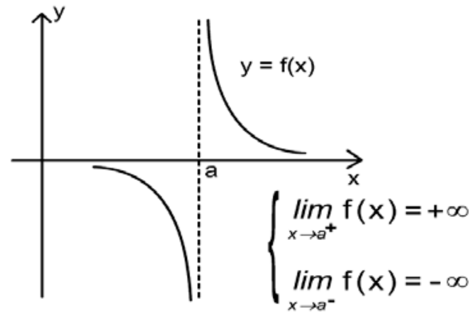


$\lim_{x\to a} f(x)=+\infty$

$\lim_{x\to a} f(x)=-\infty$







The one-sided limits are either +∞ or -∞. When we want to indicate that one of these four situations occurs without specifying which one, we shall write

Indeterminate form of type ∞ / ∞

An indeterminate form of type ∞/∞ is a limit i.e. lim f(x)/g(x) in which lim f(x) = ∞ and lim g(x) =∞ . Some examples are

$$\lim_{x \to 0^+} \frac{\ln x}{\csc x}$$

Numerator $\to -\infty$
Denominator $\to +\infty$

The following version of L'Hopital Rule is used for problems like this

THEOREM 10.3.1
(L'Hopital's Rule for Form ∞/∞)

Let lim stand for one of the limits

$$\lim_{x \to a} \quad \lim_{x \to a^+} \quad \lim_{x \to a^-} \quad \lim_{x \to +\infty} \quad \lim_{x \to -\infty}$$

and suppose that *lim* f(x) = ∞ and *lim* g(x) = ∞ .If *lim*[f′(x) / g′(x)] has a finite value L, or if this limit is +∞ or -∞ , then

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$$

EXAMPLE

Evaluate $\lim\limits_{x \to +\infty} \frac{x}{e^x}$

$$\lim_{x \to +\infty} x = +\infty \quad \text{and} \quad \lim_{x \to +\infty} e^x = +\infty$$

so that the given *limit* is an indeterminite form of type ∞/∞ .Thus , by L'Hopital's Rule

$$\lim_{x \to +\infty} \frac{x}{e^x} = \lim_{x \to +\infty} \frac{\frac{d}{dx}[x]}{\frac{d}{dx}\left[e^x\right]} = \lim_{x \to +\infty} \frac{1}{e^x} = 0$$

If $\lim f(x) = 0$ *and* $g(x) = \infty$, *then a product* limit *like*
$\lim f(x)g(x)$ *is of the type* $0.\infty$

Limit problems of this type can be converted to the form of 0/0 by writing

$$f(x)g(x) = \frac{f(x)}{1 / g(x)}$$

EXAMPLE

Evaluate $\lim\limits_{x \to 0^+} x \ln x$

Since the given problem is an indeterminite form of type $0.\infty$. We shall convert the problem to the form ∞/∞ and apply L'Hopital rule as follows:

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x}$$

$$= \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} (-x) = 0$$

Indeterminate forms of type 0^0, ∞^0 and 1^∞ and ∞-∞

Limits of the form $\lim_{x \to a} f(x)^{g(x)}$ give rise to indeterminate form of type $\mathbf{0^0}$, $\mathbf{\infty^0}$ **and** $\mathbf{1^\infty}$.

All these types are treated by first introducing a dependent variable

$$y = f(x)^{g(x)}$$

and then calculating

$$\lim_{x \to a} \ln y = \lim_{x \to a} \left[\ln\left(f(x)^{g(x)} \right) \right]$$

$$= \lim_{x \to a} \left[g(x) \ln f(x) \right]$$

once the value of $\lim_{x \to a} \ln y$ is known , it is simple to determine

$$\lim_{x \to a} y = \lim_{x \to a} f(x)^{g(x)}$$

EXAMPLE

Show that $\lim_{x \to 0} (1+x)^{1/x} = e$

Since

$\lim_{x \to 0} (1+x)^{1/x} = 1$ and $\lim_{x \to 0} 1/x = \infty$

the given limit is an indeterminite form of type 1^∞.
As discussed above , we introduce a dependent variable

$$y = (1+x)^{1/x}$$

and take the natural logarithm of both sides

$$\ln y = \ln(1+x)^{1/x} = \frac{1}{x}\ln(1+x) = \frac{\ln(1+x)}{x}$$

The limit

$$\lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{\ln(1+x)}{x}$$

is an indeterminite form of type 0/0, so by L'Hopital's Rule,

$$\lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{\ln(1+x)}{x}$$

$$= \lim_{x \to 0} \frac{1/(1+x)}{1} = 1$$

$$\lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{\ln(1+x)}{x}$$

$$= \lim_{x \to 0} \frac{1/(1+x)}{1} = 1$$

Since $\ln y \to 1$ as $x \to 0$, it follows from the continuity of the natural exponential function that $e^{\ln y} \to e^1$ or equivalent, $y \to e$ as $x \to 0$
Therefore

$$\lim_{x \to 0} (1+x)^{1/x} = e$$