Lecture # 39

Improper Integral
In this lecture we will study

- Integrals over Infinite Interval
- Integrals who's Integrands become Infinite

Integrals over Infinite Interval

As we saw before, for a given continuous function *f*, the definite integral is $\int_a^b f(x)dx$

It is assumed that the interval [a, b] is finite.

What if we look at [a, +∞) and the corresponding integral $\int_a^{+\infty} f(x)dx$

In this case, we define what is called an improper integral over an infinite interval.

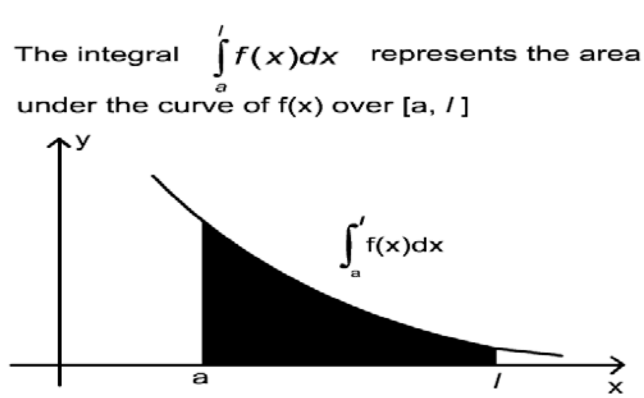
What does it mean to integrate all the way to + ∞ ?
The answer will be clear if we define this integral as a limit in the following way.

$$\int_a^{+\infty} f(x)dx = \lim_{l\to+\infty}\int_a^{l} f(x)dx$$

What this does is to first turn the integral into more familiar form of over a finite interval, and then we let the upper limits of the interval approach 0 and see what happen to the answer we had got earlier.

If this limits exists, then we say that the Improper Integral **Converges**, and the value of the limit is assigned to the integral.
If the limit does not exist, then we say that the Improper Integral **Diverges**, and no finite value is assigned.

Lets get some geometric ideas to understand things.



Evaluate $\int_1^{+\infty} \frac{dx}{x^2} = \int_1^{+\infty} \frac{1}{x^2}\, dx$

We begin by replacing the infinite upper limit with a finite upper limit l

$$\int_1^{l} \frac{dx}{x^2} = \left.\frac{-1}{x}\right]_1^{l}$$

$$= -\frac{1}{l} - (-1) = 1 - \frac{1}{l}$$

Thus

$$\int_1^{+\infty} \frac{dx}{x^2} = \lim_{l \to +\infty} \int_1^{l} \frac{dx}{x^2} = \lim_{l \to +\infty} \left(1 - \frac{1}{l}\right)$$

$$= 1 - \frac{1}{\infty} = 1 - 0 = 1$$

EXAMPLE

Evaluate $\int_1^{+\infty} \frac{dx}{x} = \int_1^{+\infty} \frac{1}{x} dx$

$$\int_1^{+\infty} \frac{dx}{x} = \lim_{l \to +\infty} \int_1^{l} \frac{dx}{x}$$

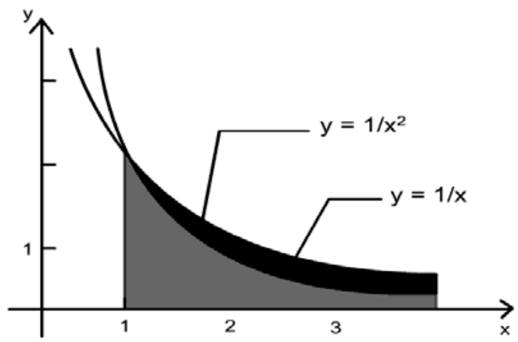
$$= \lim_{l \to +\infty} \left[\ln |x|\right]_1^{l}$$

$$= \lim_{l \to +\infty} \ln |l| = +\infty$$

What's happening here in these examples?

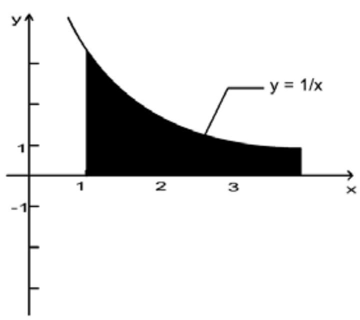
In the first case with $f(x) = 1/x^2$, we get finite answer over the same interval?
In the second case we get a divergent limit and so we were unable to calculate the area under graph of $1/x$ over $[1,+\infty)$

Look at the graphs of the two functions

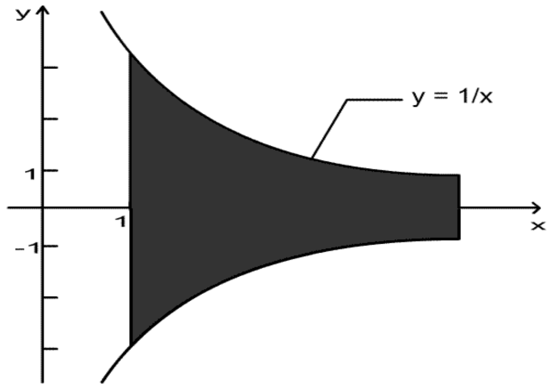


We can see geometrically that the graph of $1/x^2$ is approaching $y = 0$ much faster than that of $1/x$. Algebraically also, if you divide 1 by the square of a number, the result is much smaller than if you divide by the number itself.

For example ½ > ¼ and 1/8 > 1/64
So the idea is that as x goes to $+\infty$, $1/x^2$ goes to 0 much faster than $1/x$, so much so that when we attempt to find the area under the graph over the infinite interval $[1, +\infty)$ the first is convergent, and the other is divergent.
Lets think about Volume of second case.
Lets rotate the graphs of $1/x$ over $[1, +\infty)$ around the x-axis.



We get solid of revolution that look like funnels with no lower point as shown in figure below.



We would like to find the volume of this solid.
The cross section is a disk with radius f(x).
For f(x) = 1/x, we get for volume

$$\lim_{l\to+\infty}\int_1^l \pi\ [f(x)]^2 dx$$

$$= \lim_{l\to+\infty}\int_1^l \pi\ \left[\frac{1}{x}\right]^2 dx \ = \lim_{l\to+\infty}\int_1^l \frac{\pi}{x^2}\ dx$$

$$= \lim_{l\to+\infty}\left[-\frac{\pi}{l}+\frac{\pi}{1}\right] \quad = \pi$$

So we can find out how much paint can be held in this solid, but we cannot paint the inside of the solid!!!

We can also have an **improper integral** of this type: $\int_{-\infty}^{b} f(x)dx = \lim_{l\to-\infty}\int_{l}^{b} f(x)dx$

Integrals over Infinite Interval

We saw earlier that if a function *f* is not bounded on an interval [a, b], then *f* is not integrable on [a,b]

The integral

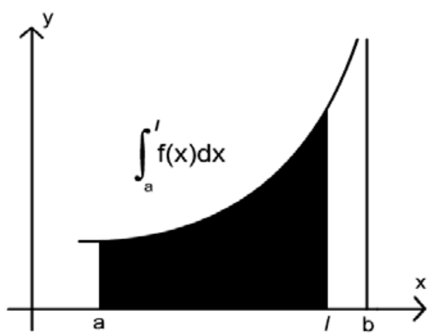
$\int_0^3 \frac{1}{(x-2)^{\frac{2}{3}}}\,dx$ is unbounded at x = 2 in [0, 3].

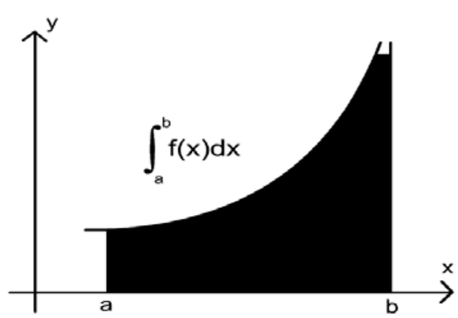
We can get around this problem by doing the following

If f is continous on [a,b) but does not have a limit from the left then we define the improper integral as a limit in this way:

$$\int_a^b f(x)dx = \lim_{l \to b^-} \int_a^l f(x)dx$$

Geometrically it can be represented as





If f is continuous on (a,b] but fails to have a limit as x approaxhes a from the right, then we define the improper integral as:

$$\int_a^b f(x)dx = \lim_{l \to a^+} \int_l^b f(x)dx$$

If f is continuous on [a, b] except that at some point c such that a<c<b, f(x) becomes infinite as x goes to c from left or right.

If both improper integrals $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ **Converge** then we say that the improper integral $\int_a^b f(x)dx$ Converges as we define $\int_a^b f(x)\, dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

EVALUATE: $\int_1^4 \frac{dx}{(x-2)^{2/3}}$

Solution: The integrand approaches $+\infty$ *as* $x \to 2$ so we solve it as

$$\int_1^4 \frac{dx}{(x-2)^{2/3}} = \int_1^2 \frac{dx}{(x-2)^{2/3}} + \int_2^4 \frac{dx}{(x-2)^{2/3}}$$

$$\int_1^4 \frac{dx}{(x-2)^{2/3}} = \lim_{l \to 2^-} \int_1^l \frac{dx}{(x-2)^{2/3}} = \lim_{l \to 2^-} \left[3(l-2)^{1/3} - 3(1-2)^{1/3}\right] = 3$$

$$\lim_{l \to 2^+} \int_l^4 \frac{dx}{(x-2)^{2/3}} = \lim_{l \to 2^+} \left[3(4-2)^{1/3} - 3(l-2)^{1/3}\right] = 3\sqrt[3]{2}$$