Lecture # 38

Work and Definite Integral

In this lecture we will discuss

- Work done by a constant force
- Work done by a variable force
- Fluid Pressure
- Pascal's Principle

Work done by a constant force

If an object moves a distance *d* along a line while a CONSTANT force F is acting on it, then work W done on the object by the force F is defined as

W = F. *d => Work = Force* x *distance*

Distance unit is meter represented by m
Force units are Pounds (lbs), Dynes or Newton (N).

One Dyne = the force needed to give a mass of 1 gram an acceleration of 1 cm/s^2

1 Newton = the force needed to give a mass of 1 Kg an acceleration of 1 m/s^2

The most common units of work are

- foot-pounds(ft lb)

- dyne-centimeters(dyne cm)

- newton-meter(N m)

One newton-meter is also called a joule(j)

Example
An object moves 5 ft along a line while subjected to a force of 100 lbs in its direction of motion. The work done is

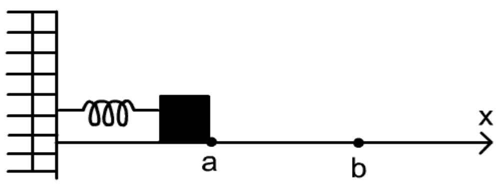
W = F.d = 100(5) = 500 ft.lbs.

Work done by a variable force

- So far the force was a constant force.
- What if it's changing constantly?
- The equation of Work we just saw will not work.
- We need Calculus

6.7.1 PROBLEM

Suppose that an object moves in the positive direction along a coordinate line while subject to a force F(x), in the direction of motion , whose magnitude depends on the coordinate x. Find the work done by the force when the object moves over an interval [a,b].



In this figure, we have a block subjected to the force of a compressed spring.
As the block moves from a to b, the spring gets un-compressed and the force it applied diminishes. So here is a case where the force varies with the position x of the spring. So the force F(x) is a function of the position of the block on the x-axis.
We need to define work done by a variable force. This, in turn, will give us the answer to calculate it as well through **definite integral.**

As before we subdivide the interval [a, b] into subintervals with coordinates
$a, x_1, ...x_{n-1}, b$ and widths Δx_1 *etc*

Work and definite integral
Let's consider the kth interval and the force F over it.

If the interval $[x_{k-1}, x_k]$ is small, then the force F will be almost constant on it.

We can approx F on this interval by F(x_k*) where x_k* is a point in the kth interval. The width of this interval will be Δx_k

So the work done on this interval is $W_k = F(x_k^*)\Delta x_k$

So the work done over the whole interval is $\sum_{k=1}^{n} W_k = \sum_{k=1}^{n} F(x_k^*)\Delta x_k$

Let the largest subinterval go to 0 to get $W = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} F(x_k^*)\Delta x_k = \int_a^b F(x)dx$

DEFINITION 6.7.2

If an object moves in the positive direction over the interval [a,b] while subjected to a variable force F(x) in the direction of motion ,then the work done by the force is

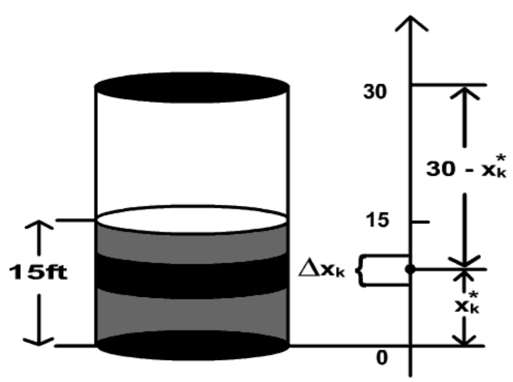
$$W = \int_a^b F(x)dx$$

Example

A cylindrical water tank of radius 10 ft and height 30 ft is half filled with water. How much force is needed to pump all the water over the upper rim of the tank?

Solution

Introduce a coordinate line as shown in figure, imagine the water to be divided into n thin layers with thicknesses $\Delta x_1, \Delta x_2, \ldots\ldots, \Delta x_n$



$(30 - x_k^*)$ represent approximate distance covered by the k^{th} layer when it moves above the rim

How much force is required to move the Kth layer of water above the rim?

The force required to move the Kth layer equals the weight of the layer, which can be found by multiplying its volume by weight density of water

$$\begin{bmatrix}\text{force to}\\ \text{move the}\\ \text{k}^{\text{th}}\text{ layer}\end{bmatrix} = (\pi r^2 \Delta x_k)\begin{bmatrix}\text{weight density}\\ \text{of water}\end{bmatrix}$$

$$= \left(\pi (10)^2 \Delta x_k\right)(62.4)$$

$$= 6240\pi \Delta x_k$$

$$W_k \approx \left(30 - x_k^*\right)6420\pi\Delta x_k$$

and the work W required to pump all layers will be approximately

$$W = \sum_{k=1}^{n} W_k \approx \sum_{k=1}^{n}\left(30 - x_k^*\right)(6420\pi)\Delta x_k$$

To find the exact value of the work we take the limit as $\max \Delta x_k \to 0$ .This yields

$$W = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n}\left(30 - x_k^*\right)(6420\pi)\Delta x_k$$

$$= \int_0^{15}(30 - x)(6420\pi)dx$$

$$= 6420\left(30x - \frac{X^2}{2}\right)_0^{15}$$

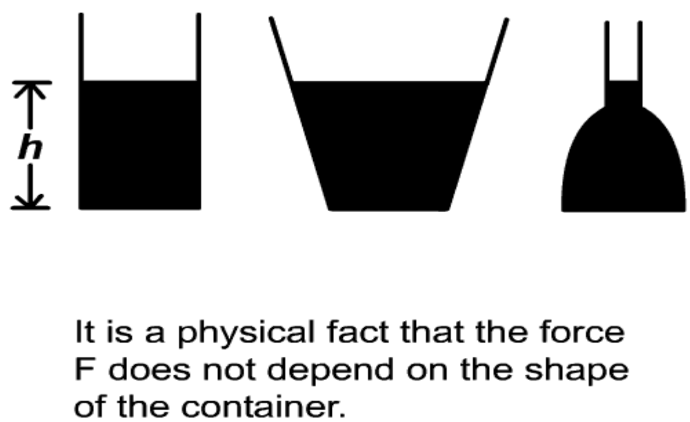
$$= 2{,}106{,}000\pi \text{ ft / lb} \approx 6{,}616{,}194 \text{ ft / lb}$$

Pressure is defined as Force per unit Area.
P = F / A = ph

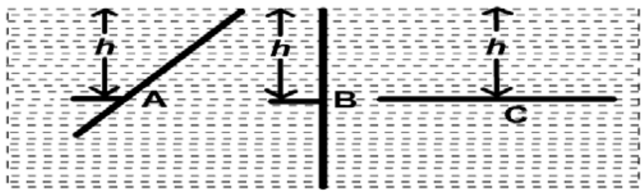
Where p is weight density and h is depth below surface of fluid.

Fluid Pressure
If a flat surface of area A is submerged horizontally in a fluid at a depth h, then the fluid exerts a force F perpendicular to the surface
This is given by $F = phA$
p is the weight density of the fluid.
p for water is 62.4 lbs/ft^3
It is a fact from Physics that the shape of the container containing the fluid does not in any way effect F.
If the three containers here have the same area for their bases and the fluid has the same height, then the F on the bases will be equal.

Figure 6.8.1

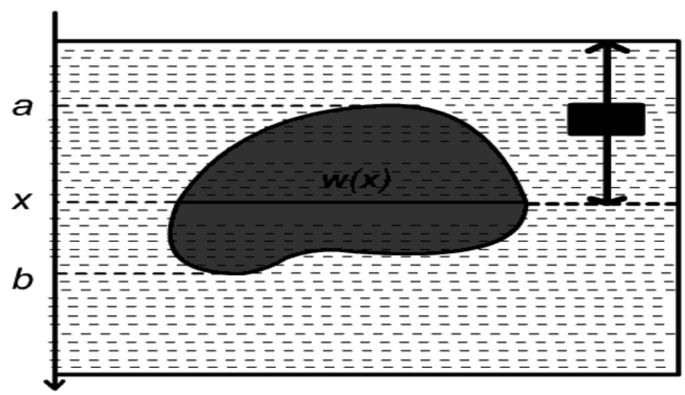


Pascal's Principle
Fluid pressure is the same in all directions at a given height.
Figure 6.8.2



By Pascal's Principle the fluid pressure
at point A ,B and C is the same

If a flat surface is submerged horizontally, then the total force on its face can be measured easily since the pressure is the same at all the points since the height of these points does not vary.
Suppose we submerge a flat surface in a fluid VERTICALLY.
Then at each point along the height of the surface, the pressure will be different and therefore Force will be different.
Here is where we need calculus
Look at the picture



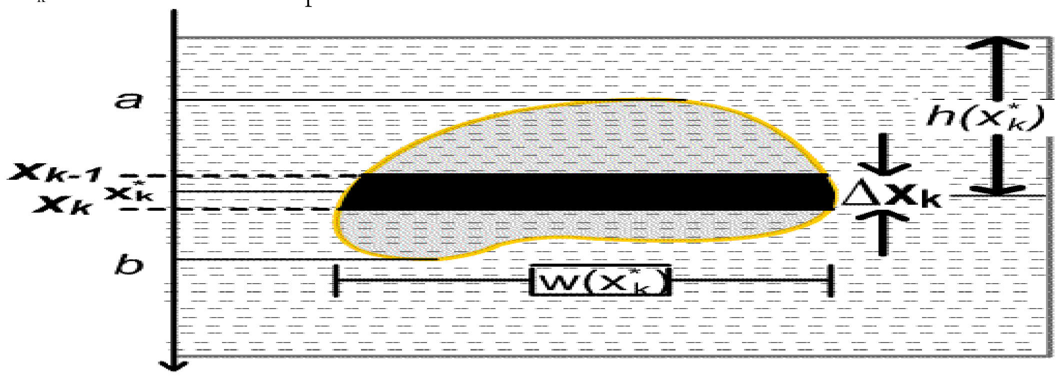
We have a surface submerged in a tank.
There is VERTICAL axis on the tank
The surface is confined by **x = a** and **x = b**
There is a height function of x that measures the depth of a point on the surface from the top of the tank.
There is a width function of x that measures the width w (x•) of the surface at the height h (x).
At different height (depth) Force on a section of the surface will be different.
We want to know the total force on the flat surface.
Subdivide the interval [a,b] into subintervals of various widths.

Using the kth interval, which form a rectangle that will be used to approximate the force on that interval on the surface?
The smaller the interval, the more accurate the approximation.
The approximation will be made by $F_k \approx ph(x_k^*)w(x_k^*)\Delta x_k$
p = fluid density

$h(x_k^*) = depth$

$w(x_k^*)\Delta x_k = area\ of\ rec\tan gle$

• F_k is the force on the strip of width almost zero on the surface



In the figure, imagine the rectangle to shrink to the dashed line.

Hence, the total force in this fashion will be $F = \sum_{k=1}^{n} F_k \approx \sum_{k=1}^{n} ph(x_k^*)w(x_k^*)\Delta x_k$

Make the largest interval go to 0 to get $F = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} ph(x_k^*)w(x_k^*)\Delta x_k = \int_a^b ph(x)w(x)dx$

Formula For Fluid Force
Assume that a flat surface is immersed vertically in a liquid of weight density P and that the submerged portion extends from x=a to x=b on a vertical x-axis.
For a<=x<=b, let w(x) be the width of the surface at x and let h(x) be the depth of the point x. Then the total **fluid pressure** on the surface is $F = \int_a^b ph(x)w(x)dx$