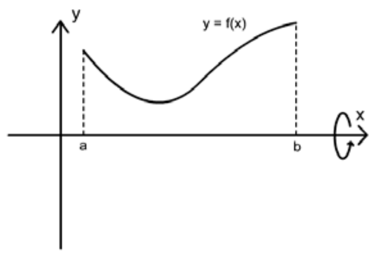
Lecture # 37
Area of a surface of Revolution

We will study in this lecture

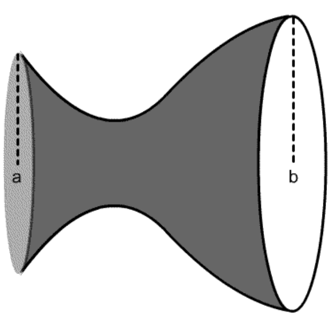
- Definition of Surface Area
- Surface Area Formulas

Surface Area Problem:

Let f be a smooth, non negative function on $[a,b]$. Find the area of the surface generated by revolving the portion of the curve $y = f(x)$ between $x = a$ and $x = b$ about the x-axis.

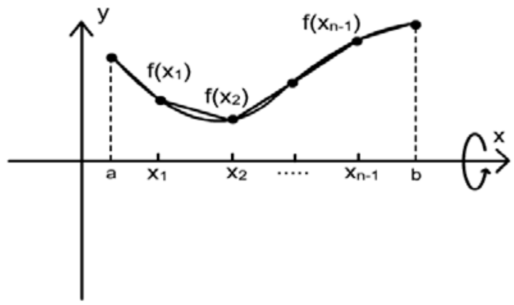


By revolving this curve we get a 3-dimensional solid as

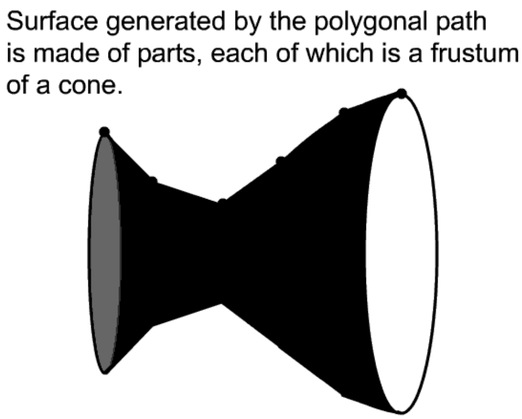


Surface area is roughly the area covered by the surface of a solid in 3-dimensional space.
But we need a more precise definition.
Say we have the graph of the function **f** as stated earlier in 6.5.1 area problem.
As usual, divide the interval [a, b] into subinterval with widths $\Delta x_1, \Delta x_2, ...\Delta x_n$ having x-coordinates $a, x_1, ...x_{n-1}, b$.
This in turn subdivides the curve into "sub-curves" which can be approximated by a polygonal path made up of line segments joining the end points of the "sub-curves"

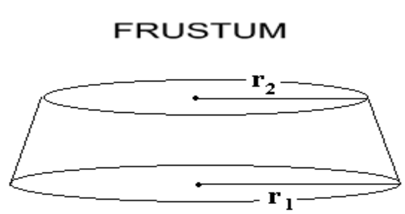
Figure 6.5.2 a)



If we now rotate this polygonal path around the x-axis, we get a solid that is approximately the same in properties to the one we had by rotating the graph of f around the x-axis.

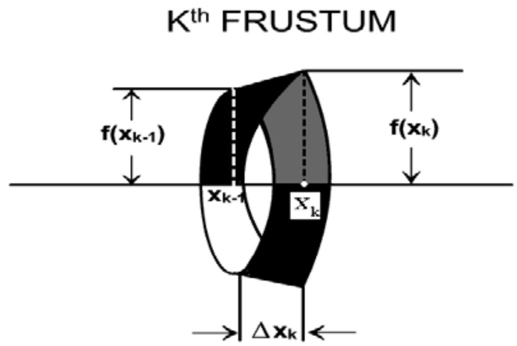


This approximated solid is made up of frustums of a cone.
A frustum of a cone is the solid that you get if you chop of the pointed top of a cone.



It has two radii and a slanted length.
Its surface area is measured by $S = \pi \ (r_1 + r_2)l$

In our case of the solid, the radii correspond to the height of the points of subdivision on the curve in an interval, and *l* is the length of the line segment (arc length) in a given interval.
So we can use this formula to find the surface area of each frustum on the approx. solid, and then add them all up.
To get the surface area of the original solid, if we let the number of our subdivisions increase without bound, then we get the surface area of the solid. Here is how it is done.
Lets consider the k-th frustum of our approx. solid.



Its surface area can be found using the formula for the frustum

$$S_k = \pi\ [f(x_{k-1}) + f(x_k)]\sqrt{(\Delta x)^2 + [f(x_k) - f(x_{k-1})]^2}$$

as $r_1 = f(x_{k-1})$

$$r_2 = f(x_k)$$

$$l = \sqrt{(\Delta x)^2 + [f(x_k) - f(x_{k-1})]^2}$$

The formula for l is just that of arc length so we can write it as $\sqrt{1 + [f'(x_k^*)]^2}\,\Delta x_k$

So

$$S_k = \pi\ [f(x_{k-1}) + f(x_k)]\sqrt{1 + [f'(x_k^*)]^2}\,\Delta x_k$$

By the Intermediate Value Theorem, there exists a point x_k^{**} in the interval $[x_{k-1}, x_k]$ such that

$$\frac{1}{2}[f(x_{k-1}) + f(x_k)] = f(x_k^{**})$$

So, we can re-write our equation for S_k as $S_k = 2\pi\ f(x_k^{**})\sqrt{1 + [f'(x_k^*)]^2}\,\Delta x_k$

The area of the approximated solid is then $\sum_{k=1}^{n} S_k = \sum_{k=1}^{n} 2\pi\, f(x_k^{**})\sqrt{1+[f'(x_k^{*})]^2}\,\Delta x_k$

As we have done before, let the largest width on the x-axis approach 0 to get

$$S = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} 2\pi\, f(x_k^{**})\sqrt{1+[f'(x_k^{*})]^2}\,\Delta x_k$$

$$S = \int_a^b 2\pi\, f(x)\sqrt{1+\left[f'(x)\right]^2}\,dx$$

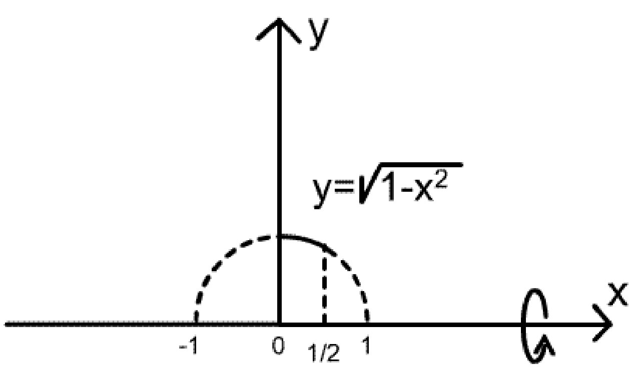
Surface Area Formulas:

Let f be a smooth, nonnegative function on $[a,b]$. Then the surface area S generated by revolving the portion of the curve $y = f(x)$ between $x = a$ and $x = b$ about the x-axis is

$$S = \int_a^b 2\pi\, f(x)\sqrt{1+\left[f'(x)\right]^2}\,dx$$

Example:
Find the surface area of the portion of the sphere generated by revolving the curve
$y = \sqrt{1-x^2} \quad 0 \le x \le \frac{1}{2}$ about the x-axis

Solution: Since



$$f(x) = \sqrt{1-x^2}$$

$$f'(x) = -\frac{x}{\sqrt{1-x^2}}$$

Thus from surface area formula

$$S = \int_0^{1/2} 2\pi\sqrt{1-x^2}\sqrt{1+\frac{x^2}{1-x^2}}dx = \int_0^{1/2} 2\pi dx = 2\pi x]_0^{1/2} = \pi$$