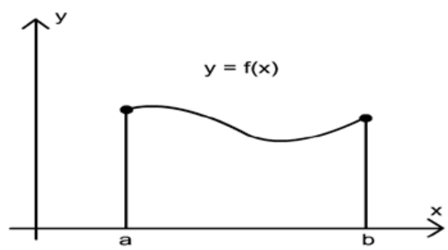
Lecture # 36
Length of a Plane Curve

Arc Length:

We all know how to find the length of a line. In this section, we will develop the ways of finding the length of curves, or lines that twist and turn. The curves we look at will be graphs of functions. The functions we look at will be such that will be continuous on a given interval. Such functions are called **smooth** functions, and their graphs **smooth curves**.
So in this lecture, we are concerned with the ARC LENGTH PROBLEM

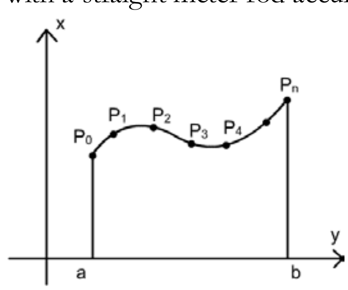
Arc Length Problem:

Suppose that f is a smooth function on the interval $[a,b]$. Find the arc length L of the curve $y = f(x)$ over the interval $[a,b]$.

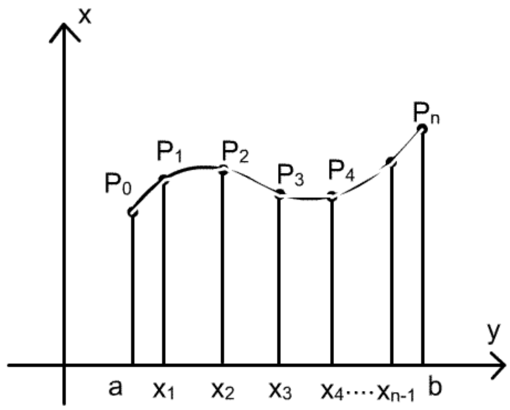


Here question arises that **How can we measure the arc length?**

Arc length is just the length of a line that is not straight, but can be rounded in various places. Sometimes we want to measure something with a measuring tape, something that is straight. But often the tape hangs loosely away from the straight object. In such cases our result is not accurate, since if the tape is loose then it will show some extra length. Similarly, if we have a curve surface, we can not measure its length with a straight meter rod accurately. Consider this curve in the figure

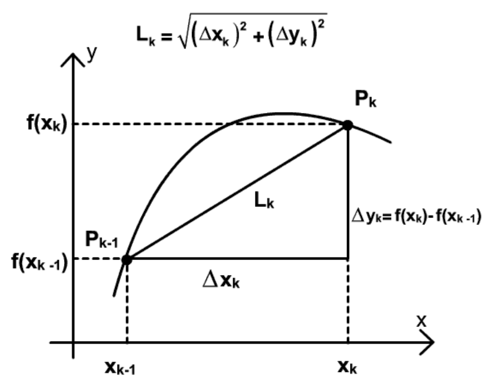


Now say I want to measure the length from P_0 to P_1. Let's say I use a straight stick. The stick will be rigid and straight, and so I will miss some length that is curved between the points P_0 and P_1. If P_0 and P_1 are so close that distance between them become straight then straight stick can be used. Now I can do the same for all the lengths between all the points in the figure.



The line segments joining the points P_i $i = (1 \text{ to } n)$ form a polygonal path and it's clear that this path is a good approximation of the length of the curve.

Well, it should also be clear that just as we have done so far, if we let the number of these segments joining all points between P_0 to P_n increase, we will get a better approximation to the curve.
We can do it in the following way
Note in the figure that to each point P_i , there corresponds a point x_i on the x-axis in the interval [a, b]
Let's assign the distance between the x_i 's as $\Delta x_1, \Delta x_2, ...\Delta x_n$.
Lets look at the k_{th} line segment. We will call it L_k. Here is what it looks like.
Figure 6.4.3



We want to find its length. Using the distance formula, we get

$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

From the figure 6.4.3, it's clear that $\Delta y_k = f(x_k) - f(x_{k-1})$

So we get $L_k = \sqrt{(\Delta x_k)^2 + (f(x_k) - f(x_{k-1}))^2}$

The Mean Value Theorem (Theorem 4.9.2) can be applied here.

$$\frac{f(x_k)-f(x_{k-1})}{x_k-x_{k-1}}=f'(x_k^*) \text{ for some point } x_k^* \text{ between } x_k \text{ and } x_{k-1}$$

$$\Rightarrow f(x_k)-f(x_{k-1})=f'(x_k^*)(x_k-x_{k-1})=f'(x_k^*)\Delta x_k$$

So, we can write our first equation as

$$L_k=\sqrt{(\Delta x_k)^2+\left(f'(x_k^*)\right)^2(\Delta x_k)^2}$$

$$L_k=\sqrt{1+\left(f'(x_k^*)\right)^2}\ \Delta x_k$$

So, this is the length of one segment.
The length of the WHOLE polygonal path will be

$$\sum_{k=1}^{n}L_k=\sum_{k=1}^{n}\sqrt{1+\left(f'(x_k^*)\right)^2}\ \Delta x_k$$

Just as we have done before, if we increase the number of divisions of the interval [a, b] in such a way that the widest of the subinterval goes to 0, we get

$L=\lim_{\max \Delta x_k\to 0}\sum_{k=1}^{n}\sqrt{1+\left(f'(x_k^*)\right)^2}\ \Delta x_k$ which we will call arc length of the curve given by the function f(x).

Of course, the above equation defines an integral $L=\int_a^b\sqrt{1+\left(f'(x)\right)^2}\ dx$

Arc Length Formulas:
If f is a smooth function on $[a,b]$, then the arc length L of the curve $y=f(x)$ from $x=a$ to $x=b$ is defined by

$$L=\int_a^b\sqrt{1+[f'(x)]^2}\,dx$$

$$=\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$