MTH101: Solution of Practice Exercise
Lecture No.14: Rate of Change

Q.No.1

Find the instantaneous rate of change of $f(x) = x^2 + 1$ at x_0.

Answer: $2x_0$

Solution:

Since $f(x) = x^2 + 1$ at x_0,

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} = \lim_{h\to 0}\frac{((x_0+h)^2+1)-(x_0^2+1)}{h},$$

$$= \lim_{h\to 0}\frac{x_0^2+h^2+2x_0h+1-x_0^2-1}{h},$$

$$= \lim_{h\to 0}\frac{h^2+2x_0h}{h},$$

$$= \lim_{h\to 0}\frac{h(h+2x_0)}{h},$$

$$= \lim_{h\to 0}(h+2x_0),$$

$$= 2x_0 \text{ by applying limit, (Answer).}$$

Q.No.2

Find the instantaneous rate of change of $f(x) = \sqrt{x+2}$ at an arbitrary point of the domain of f.

Answer: $\frac{1}{2\sqrt{a+2}}$

Solution:

Let a be any arbitrary point of the domain of f. The instantaneous rate of change of $f(x)$ at $x = a$ is

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{x\to a}\frac{\sqrt{x+2}-\sqrt{a+2}}{x-a},$$

$$= \lim_{x\to a}\frac{\sqrt{x+2}-\sqrt{a+2}}{x-a}\times\frac{\sqrt{x+2}+\sqrt{a+2}}{\sqrt{x+2}+\sqrt{a+2}} \text{ by rationalizing,}$$

$$= \lim_{x\to a}\frac{x+2-a-2}{(x-a)\sqrt{x+2}+\sqrt{a+2}},$$

$$= \lim_{x\to a} \frac{x-a}{(x-a)\,\sqrt{x+2}+\sqrt{a+2}},$$

$$= \lim_{x\to a} \frac{1}{\sqrt{x+2}+\sqrt{a+2}},$$

$$= \frac{1}{\sqrt{a+2}+\sqrt{a+2}} \text{ by applying limit,}$$

$$= \frac{1}{2\sqrt{a+2}} \quad \text{(Answer).}$$

Q.No.3

The distance traveled by an object at time t is $= f(t) = t^2$. Find the instantaneous velocity of the object at $t_0 = 4\, sec.$

Answer: 8

Solution:

$$v_{inst} = m_{tan} = \lim_{t_1\to t_0} \frac{f(t_1)-f(t_0)}{t_1-t_0},$$

$$= \lim_{t_1\to t_0} \frac{t_1^2-4^2}{t_1-t_0},$$

$$= \lim_{t_1\to t_0} \frac{t_1^2-16}{t_1-t_0},$$

$$= \lim_{t_1\to t_0} \frac{(t_1+4)(t_1-4)}{t_1-t_0},$$

$$= \lim_{t_1\to 4} \frac{(t_1+4)(t_1-4)}{(t_1-4)} \quad \text{because } t_0 = 4\, sec,$$

$$= \lim_{t_1\to 4}(t_1+4),$$

$$= 4+4 \text{ by applying limit,}$$

$$= 8 \quad \text{(Answer).}$$

Q.No.4

Find the instantaneous rate of change of $f(x) = x^3 + 1$ at $x_0 = 2$.

Answer: 12

Solution:

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h},$$

$$= \lim_{h\to 0} \frac{((2+h)^3+1)-(2^3+1)}{h},$$

$$= \lim_{h\to 0} \frac{(2^3+3(2)^2h+3(2)h^2+h^3+1)-(2^3+1)}{h},$$

$$= \lim_{h\to 0} \frac{8+12h+6h^2+h^3+1-(8+1)}{h},$$

$$= \lim_{h\to 0} \frac{9+12h+6h^2+h^3-9}{h},$$

$$= \lim_{h\to 0} \frac{12h+6h^2+h^3}{h},$$

$$= \lim_{h\to 0} \frac{h(12+6h+h^2)}{h},$$

$$= \lim_{h\to 0}(12 + 6h + h^2),$$

$$= 12 \text{ (Answer).}$$

Q.No.5

(a) The distance traveled by an object at time t is $s = f(t) = t^2$. Find the average velocity of the object between $t = 2\ sec.$ and $t = 4\ sec.$

(b) Let $f(x) = \frac{1}{x-1}$. Find the average rate of change of f over the interval [5,7].

Answer: (a) 14 **(b)** $\frac{5}{2}\ m/sec$

Solution:

(a) Avergae Velocity $= \frac{Distance\ travelled\ during\ interval}{TIme\ Elapsed},$

$$v_{ave} = \frac{f(t_1)-f(t_0)}{t_1-t_0},$$

$$= \frac{f(4)-f(2)}{4-2},$$

$$= \frac{4^2-2^2}{2},$$

$$= \frac{16-4}{2},$$

$$= \frac{12}{2},$$

$$= 6 \text{ (Answer).}$$

(b) Avergae Velocity $= \frac{Distance\ travelled\ during\ interval}{TIme\ Elapsed}$,

$$m_{sec} = \frac{f(x_1)-f(x_0)}{x_1-x_0},$$

$$= \frac{f(7)-f(5)}{7-5},$$

$$= \frac{\frac{1}{7-1}-\frac{1}{5-1}}{2},$$

$$= \frac{\frac{1}{6}-\frac{1}{4}}{2},$$

$$= -\frac{1}{24}\ m/sec.$$ (Answer).