LECTURE 9

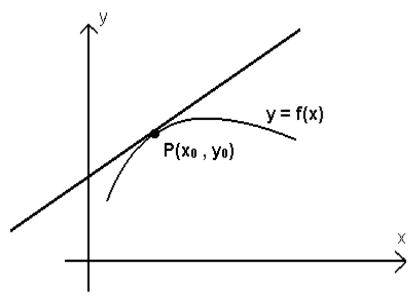
LIMITS

Calculus was motivated by the problem of finding areas of plane regions and finding tangent lines to curves. In this section we will see both these ideas

We will see how these give rise to the idea of LIMIT. We will look at it intuitively, without any mathematical proofs. These will come later.

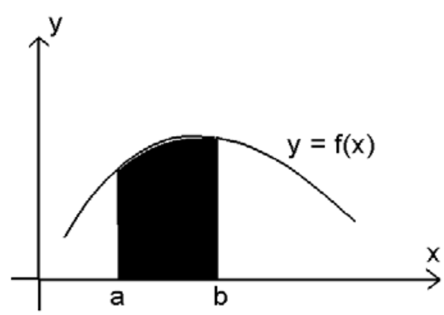
THE TANGENT PROBLEM

Given a function $f(x)$ and a point $P(x_0, y_0)$ on its graph, find an equation of the line tangent to the graph at P



AREA PROBLEM

Given a function f, find the area between the graph of f and the interval $[a, b]$ on the x-axis



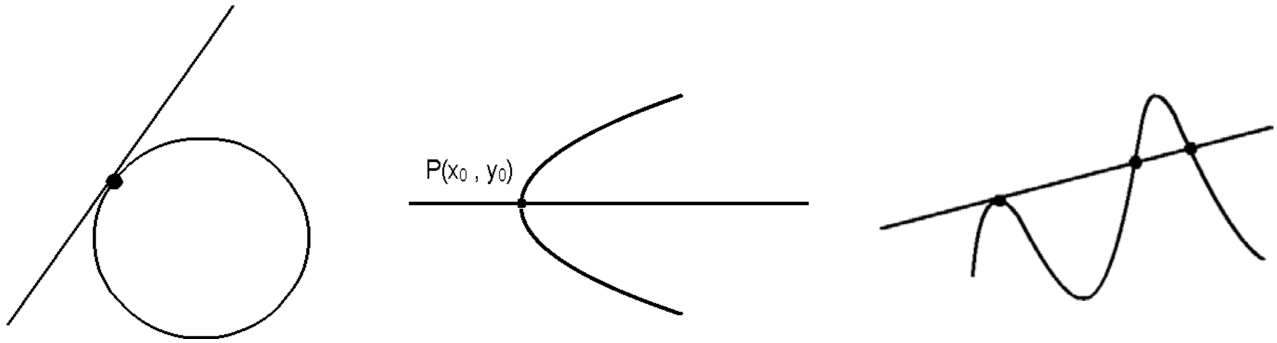
- Traditionally, the Calculus that comes out of the tangent problem is called DIFFERENTIAL CALCULUS .
- Calculus that comes out of the area problem is called INTEGRAL CALCULUS.

Both are closely related

The PRECISE definition of "tangent" and "Area" depend on a more fundamental notion of LIMIT.

Tangent Lines and LIMITS

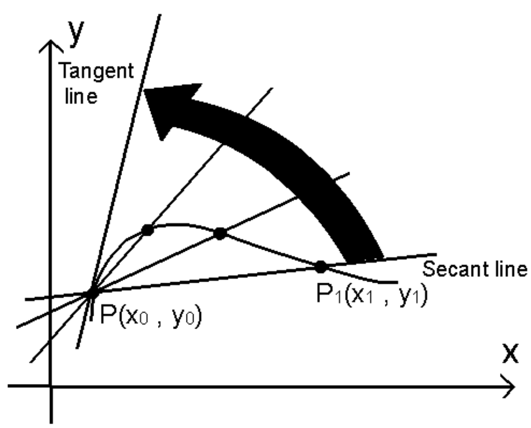
In geometry, a line is called tangent to a circle if it meets the circle at exactly one point. Figure 2.4.3a. We would like something like this to be our definition of a tangent line.
But this is not the case if you look at another curve like in figure 2.4.3b. This is a sideways parabola with a line meeting it at exactly one point. But this is not what we want as a tangent.



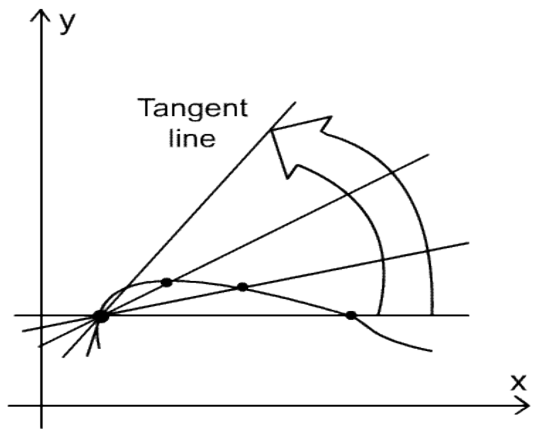
In figure 2.4.3c, we have the picture of a line that is tangent as we would like it to be, but it meets the curve more than once, and we want it to meet the curve only once.

We need to make a definition for tangent that works for all curves besides circles

Consider a point *P* on a curve in the xy-plane. Let Q be another point other than P on the curve. Draw a line through P and Q to get what is called the SECANT line for the curve. Now move the Point Q toward P. The Secant line will rotate to a "limiting" position as Q gets closer and closer to P. The line that will occupy this limiting position will be called the TANGENT line at P



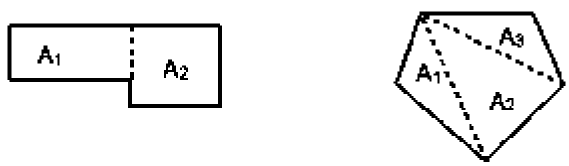
This definition works on circles too as you can see here



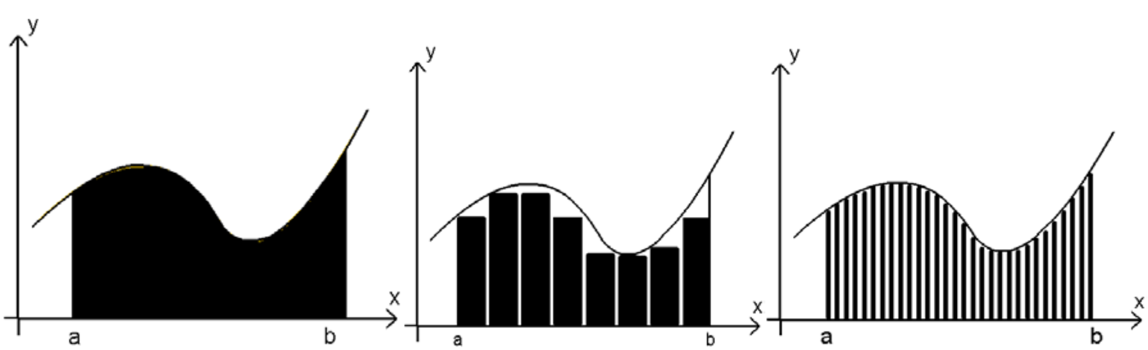
So we see how Tangents or Tangent Lines are defined using the idea of a LIMIT.

Area as a LIMIT

For most geometric shapes, the area enclosed by them can be get by subdividing the shape into finitely many rectangles and triangles. These FILL UP the shape



Some time this is not possible. Here is a regions defined in the xy-plane.



Can't be broken into rectangles and triangles that will FILL UP the area btw the curve and the interval [a,b] on the x-axis. Instead we use rectangles to APPROXIMANTE the area. Same width rects and we add their areas. If we let our rectangles increase in number, then the approx will be better and the result will be get as a

LIMITING value on the number of rects. If we let our rectangles increase in number, then the approx will be better and the result will be getting as a LIMITING value on the number of rects.

LIMITS

Let's discuss LIMIT in detail. Limits are basically a way to study the behavior of the y-values of a function in response to the x-values as they approach some number or go to infinity.

EXAMPLE

Consider

$$f(x) = \frac{\sin(x)}{x}$$ where x is in radians.

Remember that *PI* radians = 180 degrees.

f (x) is not defined at x = 0.
What happens if you get very close to x = 0??
We can get close to 0 from the left of 0, and from the right of 0.
x can approach 0 along the negative x-axis means from the left.
x can approach 0 along the positive x-axis means from the right.
From both sides we get REALLY close to 0, but not equal to it.
This getting really close is called the LIMITING process.
We write

$$\lim_{x\to 0^+} \frac{\sin(x)}{x}$$

To mean
"The limit of f (x) as x approaches 0 from the right" , the plus on the 0 stands for "from the right"
This is called the RIGHT HAND LIMIT.

$$\lim_{x\to 0^-} \frac{\sin(x)}{x}$$

To mean
"The limit of f (x) as x approaches 0 from the left" , the minus on the 0 stands for "from the left"
This is called the LEFT HAND LIMIT.
Let us see what happens to f (x) as x gets close to 0 from both right and left

X	$f(x) = \frac{\sin(x)}{x}$
1.0	0.84147
0.8	0.89670
0.6	0.94107
0.4	0.97355
0.2	0.99335
0.01	0.99998

X	$f(x) = \frac{\sin(x)}{x}$
-1.0	0.84147
-0.8	0.89670
-0.6	0.94107
-0.4	0.97355
-0.2	0.99335
-0.01	0.99998

The tables show that as x approaches 0 from both sides f(x) approaches 1, We write this as

$$\lim_{x\to 0^+}\frac{\sin(x)}{x}=1 \qquad \lim_{x\to 0^-}\frac{\sin(x)}{x}=1$$

When both the left hand and right hand limits match, we say that the LIMIT exists
We write this as

$$\lim_{x\to 0}\frac{\sin(x)}{x}=1$$

0 was a special point. In general its limit as x approaches x_0, Write this as $x \to x_0$

TABLE of Limit Notations and situations

NOTATION	HOW TO READ THE NOTATION
$\lim_{x \to x_0^+} f(x) = L_1$	The limit of f(x) as x approaches x_0 from the right is equal to L_1
$\lim_{x \to x_0^-} f(x) = L_2$	The limit of f(x) as x approaches x_0 from the left is equal to L_2
$\lim_{x \to x_0} f(x) = L$	The limit of f(x) as x approaches x_0 is equal to L

Sometimes Numerical evidence for calculating limits can mislead.

EXAMPLE

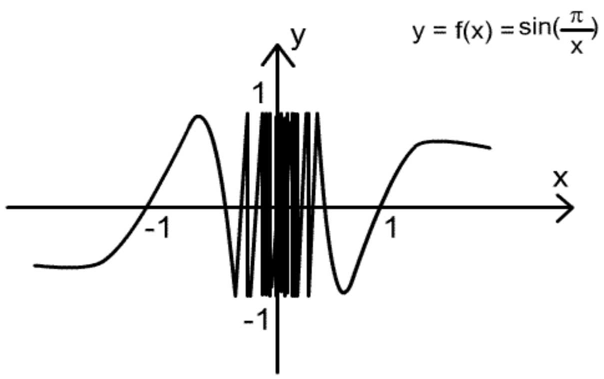
Find

$$\lim_{x\to 0}\ \sin\left(\frac{\pi}{x}\right)=\lim_{x\to 0^+}\ \sin\left(\frac{\pi}{x}\right)=\lim_{x\to 0^-}\ \sin\left(\frac{\pi}{x}\right)$$

Table Showing values of f (x) for various x

x	f(x)	x	f(x)
1	0	-1	0
0.1	0	-0.1	0
0.01	0	-0.01	0
0.001	0	-0.001	0
0.0001	0	-0.0001	0

Table suggests Limit is 0



Graph has NO LIMITING value as it OSCILLATES btw 1 and -1

Existence of Limits

Functions don’t always have a limit as the x values approach some number.

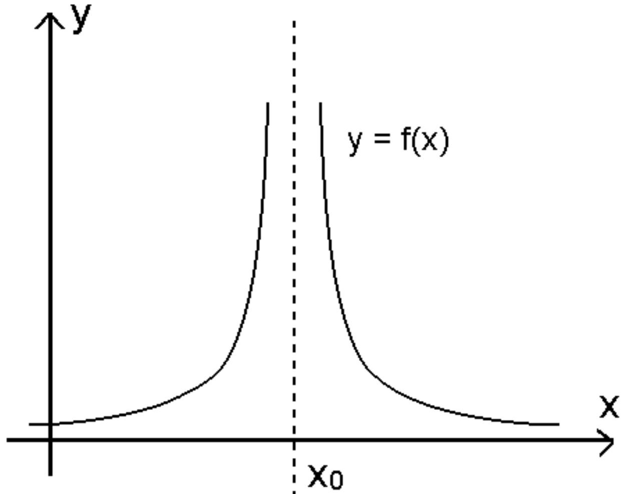
If this is the case, we say, LIMIT DOES NOT EXIST OR DNE!

Limits fail for many reason, but usual culprits are

- Oscillations
- unbounded Increase or decrease

Example

The graph of a function f(x) is given here



$$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = \lim_{x \to x_0} f(x) = +\infty$$

Note that the values of f(x) = y increase without bound as from both the left and the right. We say that

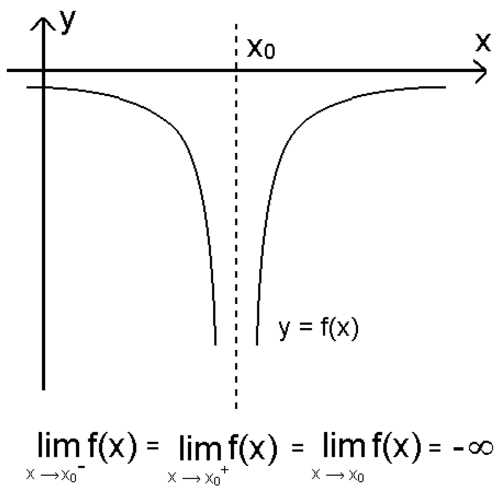
$$\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = \lim_{x \to x_0} f(x) = +\infty$$

This is a case where the LIMIT FAILS to EXIST because of unbounded-ness.

The +infinity is there to classify the DNE as caused by unbounded-ness towards +infinity. It is not a NUMBER!!

Example

The graph of a function f (x) is given here



Note that the values of f (x) = y DECREASE without bound as $x \to x_0$ from both the left and the right . We say

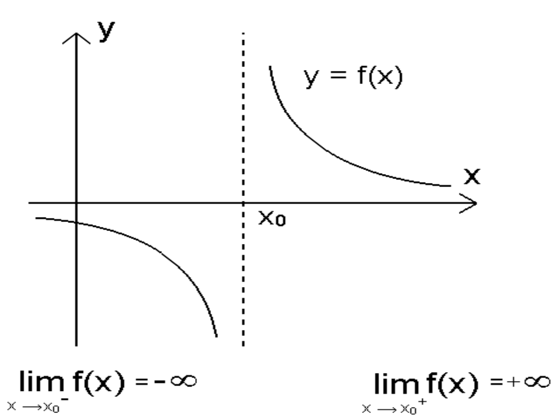
$$\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = \lim_{x \to x_0} f(x) = -\infty$$

This is a case where the LIMIT FAILS to EXIST because of unbounded-ness .

The - infinity is there to classify the DNE as caused by unbounded-ness towards – infinity. It is not a NUMBER!!

Example

Let f be a function whose graph is shown in the picture and let x approach x0, then from the picture



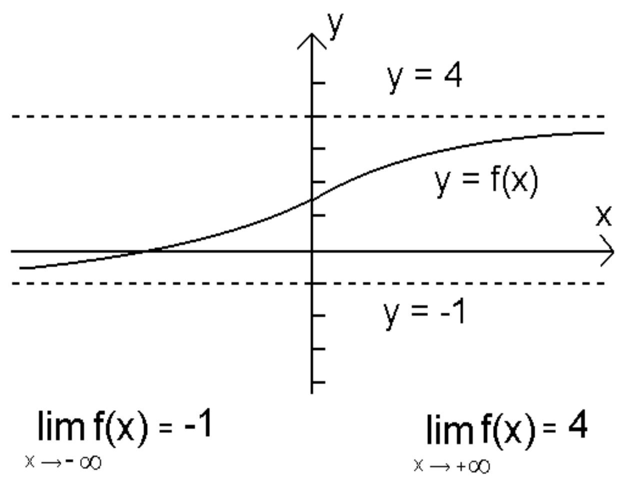
$$\lim_{x \to x_0^+} f(x) = +\infty$$

$$\lim_{x \to x_0^-} f(x) = -\infty$$

Here the two sided limits don't match up! Thus, the limit does not exist.
So far we saw limits as x approached some point x0 Now we see some limits as x goes to +inf or –inf.

Example

The graph of y = f (x) is given here We can see from it that



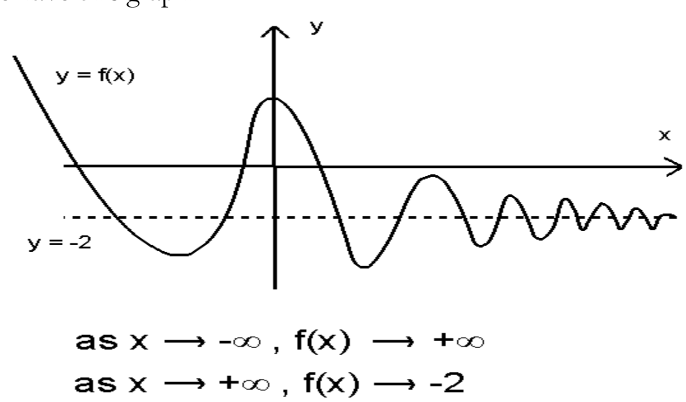
$$\lim_{x \to x_0^+} f(x) = 4$$

$$\lim_{x \to x_0^-} f(x) = -1$$

Note that when we find limits at infinity, we only do it from one side. The reason is that you can approach infinity from only one side!! x goes to infinity means that x gets bigger and bigger, and x an do that only from one side depending on whether it goes to +inf or –inf.

EXAMPLE

For this function we have this graph



Although the graph oscillates as x goes to +inf, the oscillations decreases and settle down on y = -2