Lecture 7

Operations on Functions

- Like numbers, functions can be OPERATED upon
- Functions can be added
- Functions can be subtracted
- Functions can be multiplied
- Functions can be divided
- Functions can be COMPOSED with each other

Arithmetic Operations on Functions

Just like numbers can be added etc, so can be functions

Example

$f(x) = x^2$ and $g(x) = x$

then

$f(x) + g(x) = x^2 + x$

This process defines a new function called the SUM of *f* and *g* functions We denote this SUM as follows So formally we say

$(f + g)(x) = f(x) + g(x)$

Definitions for Operations on Functions

Given functions f and g, then we define their sum, difference, product, and quotient as follows

$$(f + g)(x) = f(x) + g(x)$$
$$(f - g)(x) = f(x) - g(x)$$
$$(f.g)(x) = f(x).g(x)$$
$$(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$$

- For the function $f+g$, $f-g$, $f \,.\, g$, the domains are defined as the intersection of the domains of *f* and *g*

- For f / g, the domain is the intersection of the domains of f and g except for the points where g(x)=0

Example

$f(x) = 1 + \sqrt{x-2} \qquad g(x) = x - 1$

$$(f+g)(x) = f(x) + g(x)$$

$$= (1 + \sqrt{x-2}) + (x-1)$$

$$= x + \sqrt{x-2}$$

- Domain of f is $[2, +\infty)$
- Domain of g is $(-\infty, +\infty)$
- Domain of f +g is $[2, +\infty) \cap (-\infty, +\infty) = [2, +\infty)$

Example

$f(x) = 3\sqrt{x}$ and $g(x) = \sqrt{x}$

Find (f .g)(x)

$$(f.g)(x) = f(x).g(x) = (3\sqrt{x})(\sqrt{x})$$
$$= 3x$$

- The natural domain of 3x is (-inf, +inf).
- But this would be wrong in light of the definition of (f .g)
- By definition, the domain should be the intersection of f and g, which is [0, +inf).
- So we need to clarify that this 3x is got from a product and is different b y virtue of its domain from the standard 3x
- We do this by writing

$$(f.g)(x) = 3x \qquad x \geq 0$$

NOTATION

f multiplied by itself twice

$$f^2(x) = f(x) \cdot f(x)$$

f multiplied by itself n times

$$f^n(x) = f(x) \cdot f(x) \ldots \cdot f(x)$$

e.g. $(\sin x)^2 = \sin^2(x)$

Composition of Functions

- A new operation called COMPOSITION
- Has no analog with the arithmetic operations we saw
- Remember that the independent variable usually *x* can be given a numerical values from the domain of the function
- When two functions are composed, ONE is assigned as a VALUE to the independent variable of the other.

$f(x) = x^3$ and $g(x) = x + 4$

Compose f with g is written and defined as

$$(fog)(x) = f(g(x))$$

So

$$\begin{aligned}(fog)(x) &= f(g(x)) \\ &= (g(x))^3 \\ &= (x+4)^3\end{aligned}$$

Domain of the new function $(f \ o \ g)$ consists of all x in the domain of *g* for which *g(x)* is in the domain of *f*.

In order to compute $f(g(x))$ one needs to FIRST compute g(x) for an x from the domain of g, then

one needs g(x) in the domain of *f* to compute $f(g(x))$

Put the Sock on first, then the show

$$f(x) = x^2 + 3 \quad g(x) = \sqrt{x}$$
$$(f \circ g)(x) = f(g(x))$$
$$= (g(x))^2 + 3$$
$$= (\sqrt{x})^2 + 3 = x + 3$$

- Domain of g is $[0, +\infty)$
- Domain of f is $(-\infty, +\infty)$
- Domain of $(f \circ g)$ consists of all x in $[0, +\infty)$ such that g (x) lies in $(-\infty, +\infty)$
- So its domain is $[0, +\infty)$

$$(fog) \neq (gof)$$

Generally. Try on the last example

Put Sock on then Shoe ≠ Put Shoe on first then Sock.

- Expressing functions as a decomposition
- Sometimes want to break up functions into simpler ones
- This is like DECOMPOSING them into composition of simpler ones

$$h(x) = (x+1)^2$$

- First we add 1 to x
- Then we square x + 1
- So we can break up the function as

$$f(x) = x + 1$$
$$g(x) = x^2$$
$$h(x) = g(f(x))$$

There are more than one way to decompose a function

Example

$$(x^2+1)^{10} = [(x^2+1)^2]^5 = f(g(x)) \qquad g(x) = \left(x^2+1\right)^2 \quad f(x) = x^5$$

AND

$$(x^2+1)^{10} = [(x^2+1)^3]^{\frac{10}{3}} = f(g(x)) \qquad g(x) = \left(x^2+1\right)^5 \quad f(x) = x^{\frac{10}{3}}$$

$$T(x) = \sqrt{(\frac{x}{3})^3} = f(g(h(x)))$$

$$f(x) = \sqrt{x}$$

$$g(x) = x^3$$

$$h(x) = \frac{x}{3}$$

Here is a tables of some functions decomposed as compositions of other functions.

Function	g(x) Inside	f(x) Outside	composition
$(x^2+1)^{10}$	x^2+1	x^{10}	$(x^2+1)^{10}=f(g(x))$
$\sin^3 x$	$\sin x$	x^3	$\sin^3 x=f(g(x))$
$1/(x+1)$	$x+1$	$1/x$	$1/(x+1) = f(g(x))$
$\tan(x^5)$	x^5	$\tan x$	$\tan(x^5)=f(g(x))$

Classification of Functions

- Constant Functions

  These assign the same NUMBER to every x in the domain

  f(x) = 2 SO f(1) = 2 f(-7) = 2 etc

- Monomial in x

Anything that looks like cx^n with c a constant and n any NONEGATIVE INTEGER e.g

$$2x^5, \sqrt{3}x^{55}$$

$4x^{-4}, 5x^{\frac{3}{2}}$ Not MONOMIAL as powers are not NONEGATIVE INTEGERS.

Polynomial in x Things like

$$4x^4 + 3x^2 + 1, \quad 17 - \frac{4}{3}x^2$$

In general anything like

$$f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$$

