Lecture 6

Functions

In this lecture we will discuss

- **What a function is**
- **Notation for functions**
- **Domain of a function**
- **Range of a function**

Term function was used first by the French mathematician Leibiniz. He used it to denote the DEPENDENCE of one quantity on another.

EXAMPLE

The area A of a circle depends on its radius r by the formula

$$A = \pi r^2$$

so we say " Area is a FUNCTION of radius "

The velocity v of a ball falling freely in the earth's gravitational field increases as time t passes by. So

"velocity is a FUNCTION of time "

Function

If a quantity y depends on another quantity x in such a way that each value of x determines exactly one value of y, we say that y is a function of x.

Example

y = 4x+1 is a function. Here is a table

x	Value of y = 4x + 1
2	9
1	5
0	1
-1/4	0
$\sqrt{3}$	$4\sqrt{3} + 1$

Above table shows that each value assigned to x determines a unique value of y.

• We saw many function in the first chapter.

• All equations of lines determine a functional relationship between x and y

NOT A FUNCTION

$y = \pm\sqrt{x}$

$If\ \ x = 4\ then\ y = \pm\sqrt{4}$

thus y = +2 and y = -2

So a single value of x does not lead to exactly one value of y here. So this equation does not describe a function.

NOTATIONS FOR FUNCTIONS

In the 1700's, Swiss mathematician Euler introduced the notation which we mean y = f(x).

This is read as " y equals f of x " and it indicate that y is a function of x.

This tells right away which variable is independent and dependent.

- The one alongside the f is INDEPENDENT (usually x)
- The other one is DEPENDENT (usually y)
- f (x) is read as "y function of x" NOT AS "f multiplied by x"
- f does not represent a number in anyway. It is just for expressing functional relationship

Functions are used to describe physical phenomenon and theoretical ideas concretely.

- The idea of $A = \pi r^2$ gives us a way to express and do calculations concerning circles.
- Nice thing about this notation is that it shows which values of x is assigned to which y value.

Example

$y = f(x) = x^2$

Then $f(3) = (3)^2 = 9$

$f(-2) = (-2)^2 = 4$

Any letter can be used instead of f

y = f(x), y = g(x), y = h(x)

- Also, any other combination of letters can be used for Independent and dependant variables instead of x and y

For example s = f (t) states that the dependent variable s is a function of the independent variable t .

Example:

$$If\ \phi(x) = \frac{1}{x^3 - 1}$$

Then

$$\phi(5^{1/6}) = \frac{1}{(5^{1/6})^3 - 1} = \frac{1}{5^{3/6} - 1} = \frac{1}{\sqrt{5} - 1}$$

$$\phi(1) = \frac{1}{1-1} = \frac{1}{0} = undefined$$

So far we have used numerical values for the x variable to get an output for the y as a number. We can also replace x with another variable representing number. Here is example

If $F(x) = 2x^2-1$

Then $F(d) = 2d^2-1$

and $F(t-1) = 2(t-1)^2-1 = 2t^2-4t+2-1 = 2t^2-4t+1$

- If two functions look alike in all aspects other than a difference in g variables, then they are the SAME

  $g(c) = c^2 - 4c$ and $g(x) = x^2 - 4x$

These two are the SAME function

You can substitute a few values for c and x in the two functions and notice that the results are the same.

Formula structure matters, not the variables used.

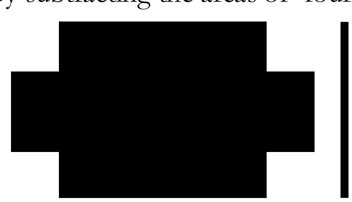
DOMAIN OF A FUNCTION

- The independent variable is not always allowed to take on any value in functions
- It may be restricted to take on values from some set.
- This set is called the DOMAIN of the function.
- It is the set consisting of all allowable values for the independent variable.
- DOMAIN is determined usually by physical constraints on the phenomenon being represented by functions.

EXAMPLE

Suppose that a square with a side of length x cm is cut from four corners of a piece of cardboard that is 10 cm square, and let y be the area of the cardboard that remain. By subtracting the areas of four corners squares from the original area, it follows that

$$y = 100 - 4x^2$$



Here x can not be negative, because it denote length and its value can not exceed 5.
Thus x must satisfy the restriction $0 \le x \le 5$. Therefore, even though it is not stated explicitly, the underlying physical meaning of x dictates that the domain of function is the set $\{0 \le x \le 5\}$ = [0,5]

We have two types of domain

1 NATURAL DOMAIN

2 RESTRICTED DOMAIN

NATURAL DOMAIN

Natural domain comes out as a result of the formula of the function. Many functions have no physical or geometric restrictions on the independent variable. However, restrictions may arise from formulas used to define such functions.

Example

$$h(x) = \frac{1}{(x-1)(x-3)}$$

- If x = 1, bottom becomes 0
- If x = 3, bottom becomes 0

So 1 and 3 is not part of the domain Thus the domain is

$$(-\infty, 1)U(1,3)U(3,+\infty)$$

If a function is defined by a formula and there is no domain explicitly stated, then it is understood that the domain consists of all real numbers for which the formula makes sense, and the function has a real value. This is called the **natural domain** of the function.

RESTRICTED DOMAINS

Sometimes domains can be altered by restricting them for various reasons. It is common procedure in algebra to simplify functions by canceling common factors in the numerator and denominator. However, the following example shows that this operation can alter the domain of a function.

Example $h(x) = \dfrac{x^2-4}{x-2}$

This function has a real value everywhere except at x=2, where we we have a division by zero. Thus the domain of **h** consists of all x except x=2. However if we rewrite it as

$$h(x) = \frac{(x-2)(x+2)}{x-2}$$
$$h(x) = (x+2)$$

Now h(x) is defined at x=2, since h(2)=2+2=4

Thus our algebraic simplification has altered the domain of the function. In order to cancel the factor and not alter the domain of h(x), we must restrict the domain and wrlte

h(x) = x+2 , x = 2

Range of a Function

- For every values given to the independent variable from the domain of a function, we get a corresponding y value .
- The set of all such y values is called the RANGE of the Function
- In other words, Range of a function is the set of all possible values for f(x) as x varies over the domain.

Techniques For Finding Range

- **By Inspection**

Example

Find the range of $f(x) = x^2$

Solution: Rewrite it as $y = x^2$

Then as x varies over the reals, y is all positive reals.

Example

Find the range of $g(x) = 2 + \sqrt{x-1}$

Solution: Since no domain is stated explicitly, the domain of g is the natural domain $[1, +\infty)$. To determine the range of the function g, let $y = 2 + \sqrt{x-1}$

As x varies over the interval $[1, +\infty)$, the value of $\sqrt{x-1}$ varies over the interval $[0, +\infty)$, so the value of $y = 2 + \sqrt{x-1}$ varies over the interval $[2, +\infty)$. This is the range of g.

- **By some algebra**

Example:

Find the range of the function $y = \frac{x+1}{x-1}$

Solution: The natural domain of x is all real numbers except 1. The set of all possible y values is not at all evident from this equation. However solving this equation for x in terms of y yields

$$x = \frac{y+1}{y-1}$$

It is now evident that y=1 is not in the range. So that range of the function is $\{y : y \neq 1\} = (-\infty, 1) U (1, +\infty)$.

Functions Defined Piecewise

Sometime the functions need to be defined by formulas that have been "pieced together".

Example

The cost of a taxicab ride in a certain metropolitan area is 1.75 rupees for any ride up to and including one mile. After one mile the rider pays an additional amount at the rate of 50 paisa per mile. If f(x) is the total cost in dollars for a ride of x miles, then the value of f(x) is

$$f(x) = \begin{cases} 1.75 & 0 < x \leq 1 \\ 1.75 + 0.50(x-1) & 1 < x \end{cases}$$

pieces have different domains

Reversing the Roles of x and y

- Usually x is independent and y dependent
- But can always reverse roles for convenience sake or other reasons whatever they maybe. For example

$$x = 4y^5 - 2y^3 + 7y - 5$$

is of the form x = g(y): that is , x is expressed as a function of y. Since it is complicated to solve it for y in terms of x, it may be desirable to leave it in this form, treating y as the independent variable and x as the dependent variable. Sometimes an equation can be solved for y as a function of x or for x as a function of y with equal simplicity. For example, the equation

$$3x + 2y = 6$$

can be written as $y = -3/2\ x + 3$ or $x = -2/3\ y + 2$

The choice of forms depends on how the equation will be used.