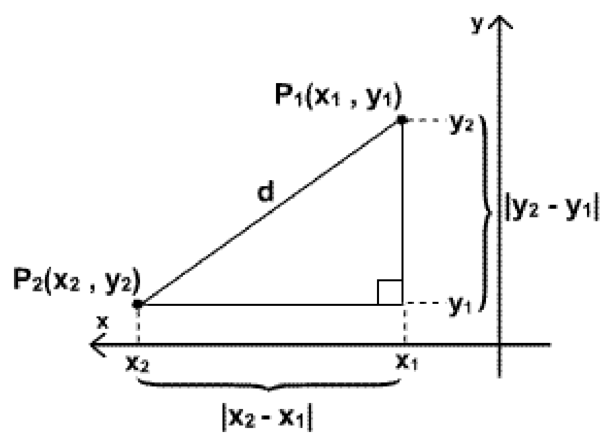
Lecture 5

Distance; Circles; Equations of the form $y = ax^2 + bx + c$

In this lecture we shall derive a formula for the distance between two points in a coordinate plane, and we shall use that formula to study equations and graphs of circles. We shall also study equations of the form $y = ax^2 + bx + c$ and their graphs.

Distance between two points in the plane

As we know that if **A** and **B** are points on a coordinate line with coordinates **a** and **b**, respectively, then the distance between **A** and **B** is **|b-a|.** We shall use this result to find the distance d between two arbitrary points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ in the plane.



If, as shown in figure, we form a right triangle With P1 and P2 as vertices, then length of the horizontal side is $|x_2-x_1|$ and the length of the vertical side is $|y_2-y_1|$, so it follows from the Pythagoras Theorem that

$$d = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2}$$

But for every real number a we have $|a|^2 = |a^2|$, thus

$$|x_2-x_1|^2 = (x_2-x_1)^2 \quad \text{and} \quad |y_2-y_1|^2 = (y_2-y_1)^2$$

Theorem 1.5.1

The distance d between two points (x_1,y_1) and (x_2,y_2) in a coordinate plane is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example

Find the distance between the points (-2,3) and (1,7).

Solution: If we let (x1,y1) be (-2,3) and let (x2,y2) be (1,7) then by distance formula we get

$$d = \sqrt{(1-(-2))^2 + (7-3)^2} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

When using distance formula, it does not matter which point is labeled (x1,y1) and which is labeled as (x2,y2). Thus in the above example, if we had let (x1,y1) be the point (1,7) and (x2,y2) the point (-2,3) we would have obtained

$$d = \sqrt{(-2-1)^2 + (3-7)^2} = \sqrt{(-3)^2 + (-4)^2} = \sqrt{25} = 5$$

which is the same result we obtained with the opposite labeling.

The distance between two points P1 and P2 in a coordinate plane is commonly denoted by d (P1,P2) or d (P2,P1).

Example

Show that the points A(4,6), B(1,-3),C(7,5) are vertices of a right triangle.

Solution: The lengths of the sides of the triangles are

$$d(A,B) = \sqrt{(1-4)^2 + (-3-6)^2} = \sqrt{9+81} = \sqrt{90}$$
$$d(A,C) = \sqrt{(7-4)^2 + (5-6)^2} = \sqrt{9+1} = \sqrt{10}$$
$$d(B,C) = \sqrt{(7-1)^2 + (5-(-3))^2} = \sqrt{36+64} = \sqrt{100} = 10$$

Since

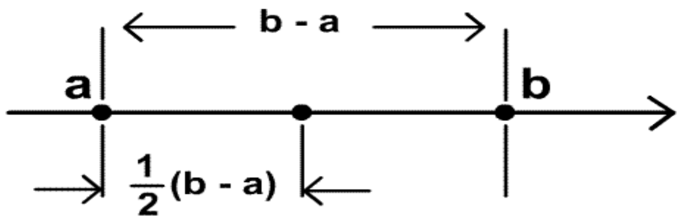
$$\left[d(A,B)\right]^2 + \left[d(A,C)\right]^2 = \left[d(B,C)\right]^2$$

It follows that ABC is a right triangle with hypotenuse BC

The Midpoint Formula

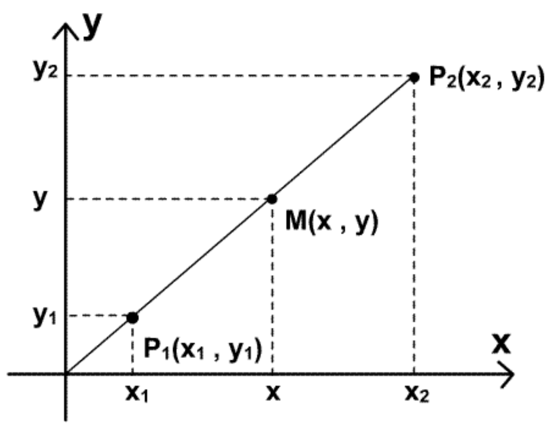
It is often necessary to find the coordinates of the midpoint of a line segment joining two points in the plane. To derive the midpoint formula, we shall start with two points on a coordinate line. If we assume that the points have coordinates **a** and **b** and that **a** **b,** then, as shown in the following figure, the distance between **a** and **b** is **b-a**, and the coordinate of the midpoint between **a** and **b** is

$$a + \frac{1}{2}(b-a) = \frac{1}{2}a + \frac{1}{2}b = \frac{1}{2}(a+b)$$



Which is the arithmetic average of **a** and **b**. Had the points been labelled with **b** $\leq$ **a**, the same formula would have resulted. Therefore, the midpoint of two points on a coordinate line is the arithmetic average of their coordinates, regardless of their relative positions. If we now let P1(x1,y1) and (x2,y2) be any two points in the plane and M(x,y) the midpoint of the line segment joining them (as shown in figure) then it can be shown using similar triangles that x is the midpoint of x1 and x2 on the x-axis and y is the midpoint of y1 and y2 on the y-axis, so

$$x = \frac{1}{2}(x_1 + x_2) \quad \text{and} \quad y = \frac{1}{2}(y_1 + y_2)$$



Thus, we have the following result.

Theorem1.5.2

(The Midpoint Formula)

The midpoint of the line segment joining two points (x1,y1) and (x2,y2) in a coordinate plane is

$$mid\ point \quad (x, y) = (\frac{1}{2}(x_1 + x_2),\ \frac{1}{2}(y_1 + y_2))$$

Example

Find the midpoint of the line segment joining (3,-4) and (7,2).

Solution: The midpoint is

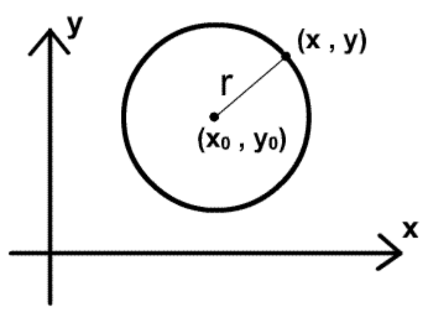
$$\left(\frac{1}{2}(3+7), \frac{1}{2}(-4+2)\right) = (5, -1)$$

Circles

If (x0,y0) is a fixed point in the plane, then the circle of radius r centered at (x0,y0) is the set of all points in the plane whose distance from (x0,y0) is r (as shown in figure). Thus, a point (x,y) will lie on this circle if and only if

$$\sqrt{(x-x_0)^2 + (y-y_0)^2} = r$$

CIRCLE



or equivalently

$$(x-x_0)^2 + (y-y_0)^2 = r^2$$

This is called the standard form of the equation of circle.

Example

Find an equation for the circle of radius 4 centered at (-5,3).

Solution: Here x0=-5, y0=3 and r=4

Substituting these values in standard equation of circle we get

$$(x-(-5))^2 + (y-3)^2 = 4^2$$
$$(x+5)^2 + (y-3)^2 = 16$$

If desired, it can be written in expanded form as

$$(x^2 + 10x + 25) + (y^2 - 6y + 9) - 16 = 0$$
$$x^2 + y^2 + 10x - 6y + 18 = 0$$

Example

Find an equation for circle with center (1,-2) that passes through (4,2)

Solution: The radius r of the circle is the distance between (4,2) and (1,-2), so

$$r = \sqrt{(1-4)^2 + (-2-2)^2} = \sqrt{(-3)^2 + (-4)^2} = \sqrt{9+16} = 5$$

We now know the center and radius, so we can write

$$(x-1)^2 + (y+2)^2 = 25 \quad \text{or} \quad x^2 + y^2 - 2x + 4y - 20 = 0$$

Finding the center and radius of a circle

When you encounter an equation of the form

$$(x-x_0)^2 + (y-y_0)^2 = r^2$$

You will know immediately that its graph is a circle; its center and radius can then be found from the constants that appear in the above equation as:

$$(x-x_0)^2 \qquad + \qquad (y-y_0)^2 \qquad = \qquad r^2$$

$(x-x_0)^2$	$(y-y_0)^2$	r^2
X-coordinate of the center is x_0	y-coordinate of the center is y_0	radius squared

Equation of a circle	center	radius
$(x-2)^2+(y-5)^2=9$	(2,5)	3
$(x+7)^2+(y+1)^2=16$	(-7,-1)	4
$x^2+y^2=25$	(0,0)	5
$(x-4)^2+y^2=5$	(4,0)	$\sqrt{5}$

The circle $x^2+y^2=1$, which is centered at the origin and has radius 1, is of special importance; it is called the **unit circle**.

Other forms for the equation of a circle

By squaring and simplifying the standard form of equation of circle, we get an equation of the form

$$x^2 + y^2 + dx + ey + f = 0$$

where d, e, and f are constants.

Another version of the equation of circle can be obtained by multiplying both sides of above equation by a nonzero constant A. This yields an equation of the form

$$Ax^2 + Ay^2 + Dx + Ey + F = 0$$

Where A, D, E and F are constants and $A \neq 0$

If the equation of a circle is given in any one of the above forms, then the center and radius can be found by first rewriting the equation in standard form, then reading off the center and radius from that equation.

Example

Find the center and radius of the circle with equation

(a) $x^2+y^2-8x+2y+8=0$ (b) $2x^2+2y^2+24x-81=0$

Solution: (a) First, group x-terms, group y-terms, and take the constant to the right side:

$(x^2-8x) + (y^2+2y) = -8$

we use completing square method to solve it as

$(x^2-8x+16) + (y^2+2y+1) = -8+16+1$

or

$(x-4)^2 + (y+1)^2 = 9$

This is standard form of equation of circle with center (4,-1) and radius 3

Solution: (b) Dividing equation through by 2 we get

$x^2+y^2+12x-81/2 = 0$

$(x^2+12x)+y^2 = 81/2$

$(x^2+12x+36)+y^2 = 81/2+36$

$$(x+6)^2 + y^2 = 153/2$$

This is standard form of equation of circle, the circle has center (-6,0) and radius $\sqrt{153/2}$

Degenerate Cases of a Circle

There is no guarantee that an equation of the form represents a circle. For example, suppose that we divide both sides of this equation by A, then complete the squares to obtain

$$(x-x_0)^2 + (y-y_0)^2 = k$$

Depending on the value of k, the following situations occur:

- (k > 0) The graph is a circle with center (x_0,y_0) and
- radius k
- (k = 0) The only solution of the equation is $x=x_0$, $y=y_0$,
- so the graph is the single point (x_0,y_0).
- (k < 0) The equation has no real solutions and

consequently no graph

Example

Describe the graphs of

(a) $(x-1)^2 + (y+4)^2 = -9$ (b) $(x-1)^2 + (y+4)^2 = 0$

Solution: (a)

There are no real values of x and y that will make the left side of the equation negative. Thus, the solution set of the equation is empty, and the equation has no graph.

Solution: (b)

The only values of x and y that will make the left side of the equation 0 are x=1, y=-4. Thus, the graph of the equation is the single point (1,-4).

Theorem

An equation of the form

$$Ax^2 + Ay^2 + Dx + Ey + F = 0$$

where $A = 0$, represents a circle, or a point, or else has no graph.

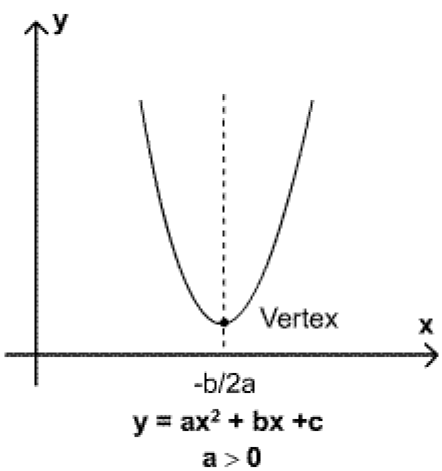
The last two cases in this theorem are called degenerate cases. In spite of the fact these degenerate cases can occur, above equation is often called the general equation of circle.

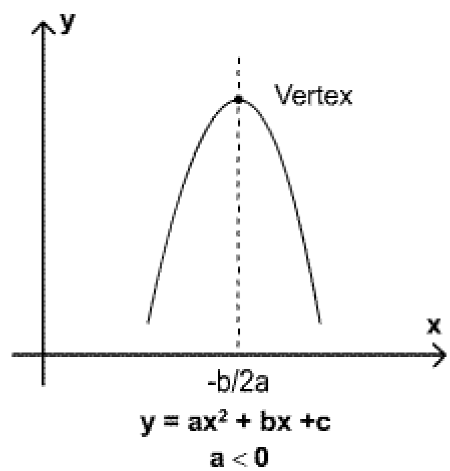
The graph of $y = ax^2 + bx + c$

An equation of the form

$$y = ax^2 + bx + c \qquad (a \neq 0)$$

Is called a quadratic equation in x. Depending on whether a is positive or negative, the graph, which is called a parabola, has one of the two forms shown below





In both cases the parabola is symmetric about a vertical line parallel to the y-axis. This line of symmetry cuts the parabola at a point called the vertex. The vertex is the low point on the curve if a>0 and the high point if a<0.

Here is an important fact. The x-coordinate of the vertex of the parabola can be found by the following formula $x = -b/2a$

Once you have the x-coordinate of the vertex, you can find the y-coordinate easily by substituting the value of x into the equation corresponding to the graph.

With the aid of this formula, a reasonably accurate graph of a quadratic equation in *x* can be obtained by plotting the vertex and two points on each side of it.

Example

Sketch the graph of

a) $y = x^2 - 2x - 2$

b) $y = -x^2 + 4x - 5$

Solution:

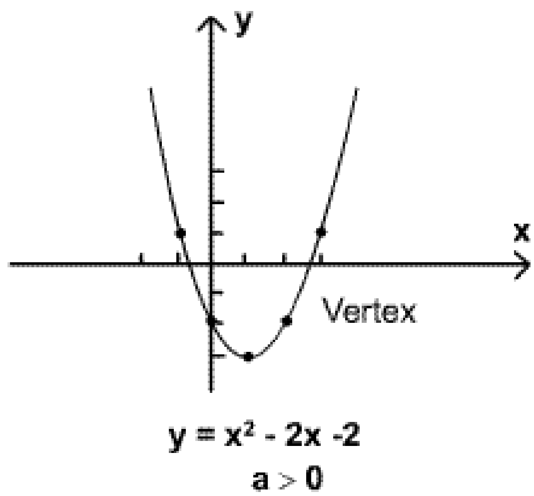
(a) This is quadratic equation with a=1, b=-2 and c=-2.

So x-coordinate of vertex is

x = - b/2a =1

Using this value and two additional values on each side as shown here

x	$y = x^2 - 2x - 2$
-1	1
0	-2
1	-3
2	-2
3	1

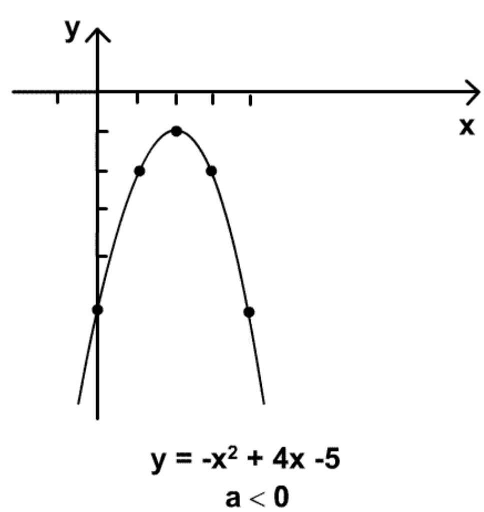


Solution: (b) This is also a quadratic equation with a=-1, b=4, and c=-5. So x-coordinate of vertex is

x = - b/2a = 2

Using this value and two additional values on each side, we obtain the table and following graph

x	$y = -x^2 + 4x - 5$
0	-5
1	-2
2	-1
3	-2
4	-5



Often, the intercepts of parabola y=ax2+bx+c are important to know. You can find y-intercept by setting x=0 and x-intercepts by setting y=0 .

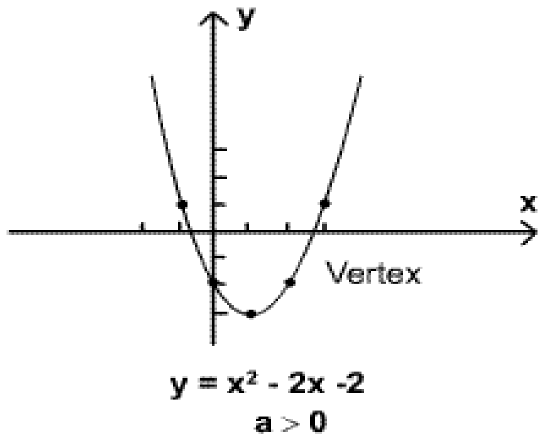
When we put y=0, then we have to solve quadratic equation $ax^2+bx+c=0$

Example

Solve the inequality $x^2-2x-2 > 0$

Solution: Because the left side of the inequality does not have discernible factors, the test point method is not convenient to use.

Instead we shall give a graphical solution. The given inequality is satisfied for those values of x where the graph of $y = x^2-2x-2$ is above the x-axis.



From the figure those are the values of x to the left of the smaller x-intercept or to the right of larger intercept.

To find these intercepts we set y=0 to obtain

$$x^2-2x-2=0$$

Solving by the quadratic formula

Gives

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{2 \pm \sqrt{12}}{2}$$

$$= 1 \pm \sqrt{3}$$

Thus, the x-intercepts are

$x = 1 + \sqrt{3}$ and $x = 1 - \sqrt{3}$

and the solution set of the inequality is

$(-\infty, 1-\sqrt{3}) \cup (1+\sqrt{3}, +\infty)$

Example

A ball is thrown straight up from the surface of the earth at time $t = 0$ sec with an initial velocity of 24.5 m/sec. If air resistance is ignored, it can be shown that the distance s (in meters) of the ball above the ground after t sec is given by

$$s = 24.5t - 4.9t^2$$

(a) Graph s versus t, making the t-axis horizontal and the s-axis vertical

(b) How high does the ball rise above the ground?

Solution (a):

The given equation is a quadratic equation with a=-4.9, b=24.5 and c=0, so t-coordinate of vertex is

t = - b/2a = - 24.5/2(-4.9) = 2.5 sec

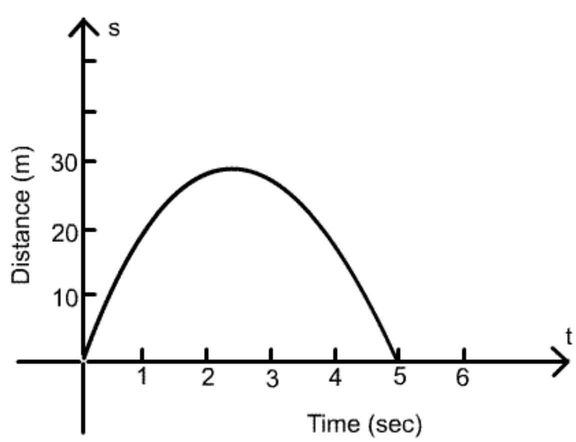
And consequently the s-coordinate of the vertex is

s = 24.5(2.5)-4.9(2.5)2 = 30.626 m

The given equation can factorize as

s = 4.9t(5-t)

so the graph has t-intercepts t=0 and t=5. From the vertex and the intercepts we obtain the graph as shown here



Solution (b) : remember that the height of the graph of a quadratic is maximum or minimum, depending on whether the graph opens UP or DOWN. From the s-coordinate of the vertex we deduce the ball rises 30.625m above the ground

Graph of $x = ay^2+by+c$

If x and y are interchanged in general quadratic equation, we get

$$x = ay^2+by+c$$

is called a quadratic in y. The graph of such an equation is a parabola with its line of symmetry parallel to the x-axis and its vertex at the point with y-coordinate y = -b