MTH101 Solution: Practice Questions
Lecture No. 44 to 45

Question 1: Find the radius of convergence for the following power series:

$$\sum_{1}^{\infty} \frac{n!\,.\,x^n}{(3n)!}.$$

Solution

Here $a_n = \dfrac{n!\,.\,x^n}{(3n)!}$,

so $a_{n+1} = \dfrac{(n+1)!\,.\,x^{n+1}}{(3n+3)!}$.

Now,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,.\,x^{n+1}}{(3n+3)!}\cdot\frac{(3n)!}{n!\,.\,x^n}\right|,$$

$$= \lim_{n\to\infty}\frac{(n+1)n!}{(3n+3)(3n+2)(3n+1)(3n)!}\cdot\frac{(3n)!}{n!}|x|,$$

$$= \lim_{n\to\infty}\frac{n+1}{(3n+3)(3n+2)(3n+1)}|x|,$$

$$= 0.$$

Thus the series converges absolutely $\forall\ x$ and radius of convergence $=\infty$.

Question 2: Show that $\sum_{1}^{\infty}|a_n|$ is divergent for the following alternating series:

$$\sum_{1}^{\infty}\frac{(-1)^n\,.\,n^n}{2n!}.$$

Solution:

Here $a_n = \dfrac{(-1)^n \,.\, n^n}{2n!}$,

so $|a_n| = \dfrac{n^n}{2n!}$,

and $|a_{n+1}| = \dfrac{(n+1)^{n+1}}{2(n+1)!}$.

Now $\lim\limits_{n\to\infty} \dfrac{|a_{n+1}|}{|a_n|} = \lim\limits_{n\to\infty} \dfrac{(n+1)^{n+1}}{2(n+1)!} \,.\, \dfrac{2n!}{n^n}$,

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n,$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n,$$

$$= e > 1.$$

Thus the given series is diverges.

Question 3: Find the first two terms of Tylor series for $f(x) = \ln x$ at $x = 2$.

Solution:

$\because \quad f(x) = \ln x, \qquad f(2) = \ln 2,$

$\Rightarrow \quad f'(x) = \dfrac{1}{x}, \qquad f'(2) = \dfrac{1}{2}.$

$\because$ Taylor polynomial for f about $x = a$:

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

$\Rightarrow p_0(x) = f(2) = \ln 2,$

$\Rightarrow p_1(x) = f(2) + f'(2)(x-2) = \ln 2 + \dfrac{1}{2}(x-2).$

Question 4: Find the first four terms of the Taylor series generated by f at $x = 2$ where

$$f(x) = \frac{1}{x+1}.$$

Solution:

$$\because \quad f(x)=\frac{1}{x+1}=\left(x+1\right)^{-1} \quad \text{and} \quad f(2)=\frac{1}{2+1}=\frac{1}{3},$$

$$\Rightarrow \quad f'(x)=-\left(x+1\right)^{-2}=-\frac{1}{\left(x+1\right)^{2}} \quad \text{and} \quad f'(2)=-\frac{1}{\left(2+1\right)^{2}}=-\frac{1}{3^{2}},$$

$$\Rightarrow \quad f''(x)=2\left(x+1\right)^{-3}=\frac{2}{\left(x+1\right)^{3}} \quad \text{and} \quad f''(2)=\frac{2}{\left(2+1\right)^{3}}=\frac{2}{3^{3}},$$

$$\Rightarrow \quad f'''(x)=-6\left(x+1\right)^{-4}=-\frac{6}{\left(x+1\right)^{4}} \quad \text{and} \quad f'''(2)=-\frac{6}{\left(2+1\right)^{4}}=-\frac{6}{3^{4}}.$$

$\because$ Taylor Series

$$p_n(x)=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\ldots$$

$$=\frac{1}{3}+\frac{-\frac{1}{3^2}}{1!}(x-2)+\frac{\frac{2}{3^3}}{2!}(x-2)^2+\frac{-\frac{6}{3^4}}{3!}(x-2)^3+\ldots$$

$$\Rightarrow \frac{1}{x+1}=\frac{1}{3}-\frac{1}{9}(x-2)+\frac{1}{27}(x-2)^2-\frac{1}{81}(x-2)^3+\ldots$$