Lecture # 42

Infinite Series

Definition 11.3.1
An Infinite series is an expansion that can be written in the form

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + \ldots + u_k + \ldots$$

The numbers u_1, u_2, … are called the terms of the series.

What does it mean to add infinitely many terms?
Can we add 1+2+3+4+…. All the way to infinity?

This is physically and mentally impossible!
But we can compute an infinite sum using the idea of limits
Look at this decimal expansion

0.333333….

We can rewrite this as an infinite series

0.3+0.03+0.003+0.0003+…
Or as

$$\frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \ldots$$

Note that we already know that 0.333…. is really 1/3

So our sum of the above series should also be 1/3

Let's work out a definition for the sum of an infinite series. Consider the following sequences of (finite) sums

$$s_1 = \frac{3}{10} = 0.3$$

$$s_2 = \frac{3}{10} + \frac{3}{10^2} = 0.33$$

$$s_3 = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} = 0.333$$

The sequence of numbers s_1,s_2, etc. can be viewed as a succession of approximations to the "sum " of the infinite series. We know that the answer is 1/3, so it is better be that the approximations are close to this at the start, and get closer and closer to this number as more and more terms are added together. This suggests that if we take the limit of the sequence of approximations, we should get 1/3
Note that the nth term of the approximating series is

$$s_n = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \ldots \frac{3}{10^n}$$

Taking its limit gives (limit of the nth partial sum of the series)

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\frac{3}{10}+\frac{3}{10^2}+\frac{3}{10^3}+\ldots\frac{3}{10^n}\right)$$

Now we do some algebraic manipulation here

$$S_n = \frac{3}{10}+\frac{3}{10^2}+\ldots+\frac{3}{10^n} \quad \ldots\ldots(A)$$

$$\lim_{n\to+\infty} S_n = \lim_{n\to+\infty}\left(\frac{3}{10}+\frac{3}{10^2}+\ldots+\frac{3}{10^n}\right)$$

$$\frac{1}{10}S_n = \frac{3}{10^2}+\frac{3}{10^3}+\ldots\ldots\ldots+\frac{3}{10^n}+\frac{3}{10^{n+1}} \quad \ldots\ldots(B)$$

Subtracting (B) from (A)

$$S_n - \frac{1}{10}S_n = \frac{3}{10}-\frac{3}{10^{n+1}}$$

$$\frac{9}{10}S_n = \frac{3}{10}\left(1-\frac{1}{10^n}\right)$$

$$S_n = \frac{1}{3}\left(1-\frac{1}{10^n}\right)$$

Since $1/10^n \to 0$ as $n \to +\infty$, it follows that

$$\lim_{n\to+\infty} S_n = \lim_{n\to+\infty}\frac{1}{3}\left(1-\frac{1}{10^n}\right) = \frac{1}{3}$$

Now we can formalize the idea of a sum of an infinite series

DEFINITION 11.3.2

Let $[S_n]$ be the sequence of partial sums of the series $u_1 + u_2 + u_3 + \ldots + u_k + \ldots$. If the sequence $[S_n]$ converges to a limit S, then the series is said to converge, and S is called the **sum** of the series. We denote this by writing

$$S = \sum_{k=1}^{\infty} u_k$$

If the sequence of partial sums diverges, then the series is said to diverge: A divergent series has no sum.

Example

Determine whether the series

$$1 - 1 + 1 - 1 + 1 - 1 + \ldots$$

converges or diverges. If it converges, find the sum.

$s_1 = 1$
$s_2 = 1 - 1 = 0$
$s_3 = 1 - 1 + 1 = 1$
$s_4 = 1 - 1 + 1 - 1 = 0$

and so forth. Thus the sequence of partial sum is

$$1, 0, 1, 0, 1, 0, \ldots$$

Since this is a divergent sequence, the given series diverges and consequently has no sum.

Geometric Series

A geometric series is one of the form

$a + ar + ar^2 + ... + ar^{k-1} + ..$

Each term is obtained by multiplying the previous one by a constant number, *r*, and this number is called the ratio for the series.

Examples are

1+2+4+8+…

1+1+1+1+…

THEOREM 11.3.3

A geometric series

$$a + ar + ar^2 + ... + ar^{k-1} + ... \quad (a \neq 0)$$

converges if $|r| < 1$ and diverges if $|r| \geq 1$. If the series converges, then the sum is

$$\frac{a}{1-r} = a + ar + ar^2 + ... + ar^{k-1} + ...$$

Example

The series

$$5 + \frac{5}{4} + \frac{5}{4^2} + ... + \frac{5}{4^{k-1}} + ...$$

is a geometric series with a = 5 and r = 1/4. Since |r| = 1/4 < 1, the series converges and the sum is

$$\frac{a}{1-r} = \frac{5}{1-1/4} = \frac{20}{3}$$

Harmonic Series
A Harmonic series is of the type

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots$$

It may look like this series converges as each successive term is smaller than the first, but in fact this series diverges!

Convergence Tests
Here are some tests that are used to determine if a series converges or diverges. These tests are applied on the kth term of a series (general term), and not on the nth partial sum as we did earlier to derive the definition of the sum of an infinite series.

THEOREM 11.4.1
(The Divergence Test)

a) If $\lim_{k\to+\infty} u_k \neq 0$, then the series $\sum u_k$ ***diverges***

b) If $\lim_{k\to+\infty} u_k = 0$, then the series $\sum u_k$ may either ***converge*** or ***diverge***.

This theorem tells us when a series diverges
The alternative form of part (a) is sufficiently important that we state it separately

THEOREM 11.4.2
If the series $\sum u_k$ converges, then

$$\lim_{k\to+\infty} u_k = 0$$

Example

The series

$$\sum_{k=1}^{\infty} \frac{k}{k+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \ldots\ldots + \frac{k}{k+1} + \ldots.$$

Diverges since

$$\lim_{k \to \infty} \frac{k}{k+1} = \lim_{k \to \infty} \frac{1}{1 + \frac{1}{k}} = 1 \neq 0$$

Algebraic Properties of Infinite Series

THEOREM 11.4.3

a) If $\sum u_k$ and $\sum v_k$ are convergent series, then $\sum (u_k + v_k)$ and $\sum (u_k - v_k)$ are convergent series and the sums of these series are related by

$$\sum_{k=1}^{\infty} (u_k + v_k) = \sum_{k=1}^{\infty} u_k + \sum_{k=1}^{\infty} v_k$$

$$\sum_{k=1}^{\infty} (u_k - v_k) = \sum_{k=1}^{\infty} u_k - \sum_{k=1}^{\infty} v_k$$

b) If c is a nonzero constant , then the series $\sum u_k$ and $\sum cu_k$ both *converge* or both *diverge*. In this case of *convergence*, the sums are related by

$$\sum_{k=1}^{\infty} cu_k = c\sum_{k=1}^{\infty} u_k$$

c) Convergence or divergence is unaffected by deleting a finite number of terms from a series; in particular, for any positive integer K, the series

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \ldots\ldots$$

$$\sum_{k=K}^{\infty} u_k = u_K + u_{K+1} + u_{K+2} + \ldots\ldots$$

both converge or both diverge.

Example

Find the sum of the series

$$\sum_{k=1}^{\infty}\left(\frac{3}{4^k} - \frac{2}{5^{k-1}}\right)$$

Solution: The series

$$\sum_{k=1}^{\infty}\frac{3}{4^k} = \frac{3}{4} + \frac{3}{4^2} + \frac{3}{4^3} + \ldots\ldots\ldots\ldots$$

is a convergent series (a = ¾, r = ¼) and the series

$$\sum_{k=1}^{\infty}\frac{2}{5^{k-1}} = 2 + \frac{2}{5} + \frac{2}{5^2} + \frac{2}{5^3} + \ldots\ldots\ldots\ldots$$

is also a convergent series (a = 2, r = 1/5).

Thus, from Theorem 11.4.3(a) and 11.3.3 the given series converges and

$$\sum_{k=1}^{\infty}\frac{3}{4^k} - \sum_{k=1}^{\infty}\frac{2}{5^{k-1}} = \frac{3/4}{1-1/4} - \frac{2}{1-1/5} = -\frac{3}{2}$$

THEOREM 11.4.4

(The Integral Test)

Let $\sum u_k$ be a series with positive terms, and let f(x) be the function that results when k is replaced by x in the formula for u_k.

If f is decreasing and continuous on

the interval $[a, +\infty]$, then

$$\sum_{k=1}^{\infty} u_k \text{ and } \int_a^{\infty} f(x)dx$$

both converge or both diverge.

This theorem allows us to study the convergence of a series by studying a related integral.

This integral is the improper integral over the interval [1, +∞) and the function is the kth term of the series.

It is interesting that the integral represents a continuous phenomenon, while the summation a discrete one!

EXAMPLE

Use the integral test to determine whether the following series converge or diverge

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

If we replace k by x in the general term 1/k, we obtain the function f(x) = 1/x.

Since

$$\int_1^{+\infty} \frac{1}{x} dx = \lim_{l \to +\infty} \int_1^{l} \frac{1}{x} dx$$

$$= \lim_{l \to +\infty} \left[\ln l - \ln 1 \right] = +\infty$$

the integral diverges and consequently so does the series

p-Series

***p*-series** or **hyper-harmonic series** is an infinite series of the form.

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots$$

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \dots \qquad where\ p = 1$$

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots \qquad where\ p = \frac{1}{2}$$

The following theorem tells us whether a p-series converges or diverges

THEOREM 11.4.5
(Convergence of p-series)

$$\sum_{k=1}^{\infty}\frac{1}{k^p}=1+\frac{1}{2^p}+\frac{1}{3^p}+......+\frac{1}{k^p}+......$$

converges if $p>1$ and diverges if $0<p\leq 1$

EXAMPLE

$$1+\frac{1}{\sqrt[3]{2}}+\frac{1}{\sqrt[3]{3}}+.......+\frac{1}{\sqrt[3]{k}}+.......$$

Since $p=\frac{1}{3}<1$ so p-series diverges