Lecture 4

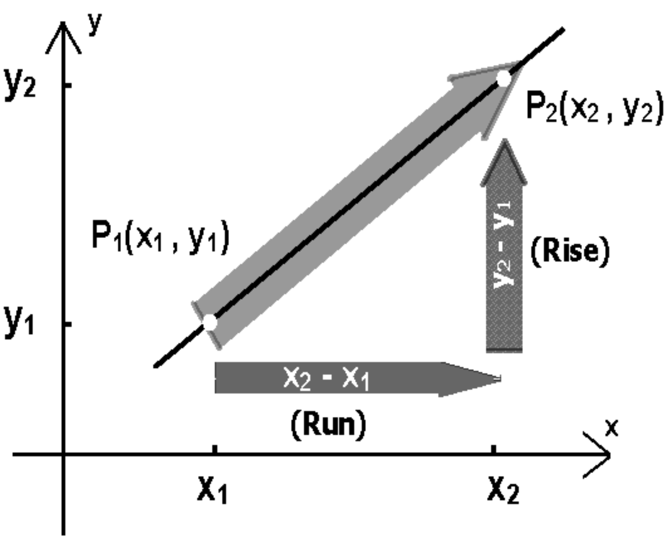
Lines

In this section we shall discuss ways to measure the "steepness" or "slope" of a line in the plane. The ideas we develop here will be important when we discuss equations and graphs of straight lines. We will assume that you have sufficient understanding of trigonometry.

Slope

In surveying, slope of a hill is defined to be the ratio of its rise to its run. We shall now show how the surveyor's notion of slope can be adapted to measure the steepness of a line in the xy-plane.

Consider a particle moving left to right along a non vertical line segment from a point P1(x1,y1) to a point P2(x2,y2). As shown in the figure below,



the particle moves y2-y1 units in the y-direction as it travels x2-x1 units in the positive x-direction. The vertical change y2-y1 is called the rise, and the horizontal change x2-x1 the run. By analogy with the surveyor's notion of slope we make the following definition.

Definition 1.4.1

If P1(x1,y1) and P2(x2,y2) are points on a non-vertical line, then the slope m of the line is defined by

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

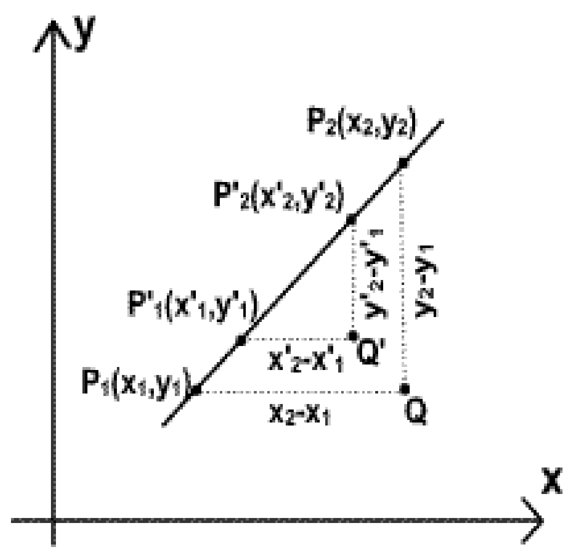
So the slope is the ratio of the vertical distance and the horizontal distance between two points on a line. We make several observations about Definition 1.4.1.

Definition 1.4.1 does not apply to vertical lines. For such lines we would have

$x_2 = x_1$

so (1) would involve a division by zero. The slope of a vertical line is UNDEFINED. Speaking informally, some people say that a vertical line has infinite slope . When using formula in the definition to calculate the slope of a line through two points, it does not matter which point is called P1 and which one is called P2, since reversing the points reverses the sign of both the numerator and denominator of (1), and hence has no effect on the ratio. Any two distinct points on a non-vertical line can be used to calculate the slope of the line that is, the slope m computed from any other pair of distinct points P1 and P2 on the line will be the same as the slope m' computed from any other pair of distinct points P'1 and P'2 on the line. All this is shown in figure below

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y'_2 - y'_1}{x'_2 - x'_1} = m'$$



Example

In each part find the slope of the line through

(a) the points (6,2) and (9,8) (b) the points (2,9) and (4,3)

(c) the points (-2,7) and (5,7)

Solution:

a) $m = \dfrac{8-2}{9-6} = \dfrac{6}{3} = 2$

b) $m = \dfrac{3-9}{4-2} = \dfrac{-6}{2} = -3$

c) $m = \dfrac{7-7}{5-(-2)} = \dfrac{0}{7} = 0$

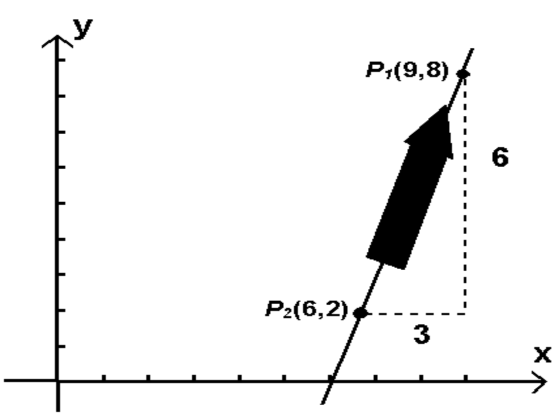
Interpretation of slope

Since the slope m of a line is the rise divided by the run, it follows that

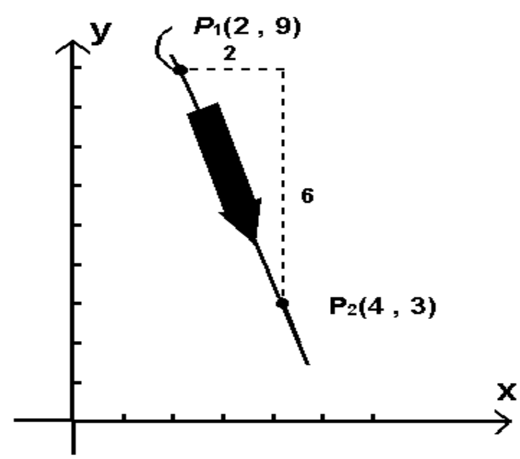
rise = m . run

so that as a point travels left to right along the line, there are m units of rise for each unit of run. But the rise is the change in y value of the point and the run is the change in the x value, so that the slope m is sometimes called the **rate of change of y with respect to x** along the line.

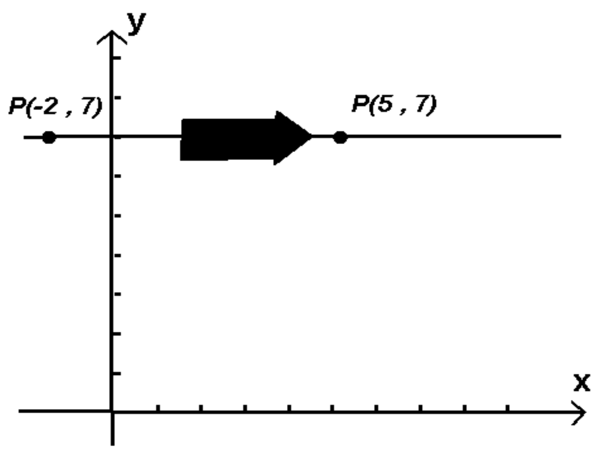
As illustrated in the last example, the slope of a line can be positive, negative or zero.

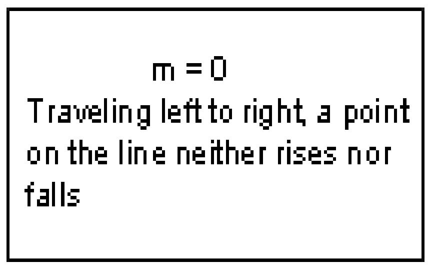


m=2
Traveling left to right, a point on the line rises two units for each unit it moves in the positive x-direction.



m = -3
Traveling left to right, a point on the line falls three units for each unit it moves in the positive x-direction.





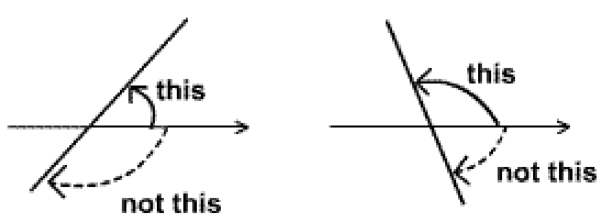
A positive slope means that the line is inclined upward to the right, a negative slope means that it is inclined downward to the right, and a zero slope means that the line is horizontal.

Angle of Inclination

If equal scales are used on the coordinate axis, then the slope of a line is related to the angle the line makes with the positive x-axis.

Definition 1.4.2

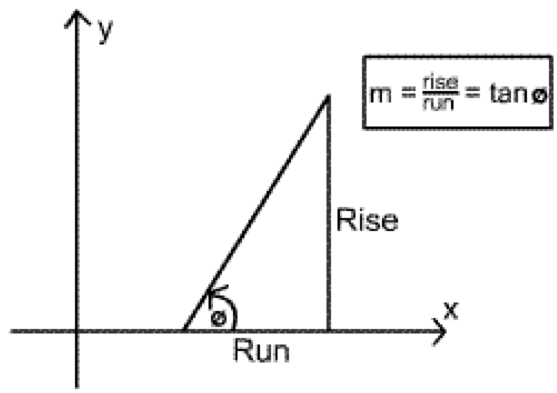
For a line L not parallel to the x-axis, the angle of inclination is the smallest angle measured counterclockwise from the direction of the positive x-axis to L (shown in figure below). For a line parallel to the x-axis, we take $\varnothing$ = 0.



Angles of inclination are measured counterclockwise from the x-axis

In degree measure the angle of inclination satisfies $0^0 \le \phi \le 180^0$ and in radian measure it satisfies $0 \le \phi \le \pi$.

The following theorem, suggested



by the figure at right, relates the

Slope of a line to its angle of Inclination.

Theorem 1.4.3

For a nonvertical line, the slope m and angle of inclination $\varnothing$ are related by

$$m = \tan\varnothing \qquad (2)$$

If the line L is parallel to the *y*-axis, then $\varnothing = \frac{1}{2}\pi$

so $\tan\varnothing$ is undefined. This agrees with the fact that the slope m is undefined for vertical lines.

Example

Find the angle of inclination for a line of slope m = 1 and also for a line of slope m = -1.

Solution : If m = 1, then from (2)

$$\tan\varnothing = 1$$

$$\varnothing = \frac{1}{4}\pi$$

If m = -1, then from (2)

$\tan\varnothing = -1$, from this equality and the fact that $0 \le \varnothing \le \pi$ we obtain

$$\varnothing = \frac{3\pi}{4}.$$

Parallel and Perpendicular Lines

Theorem 1.4.4

Let L1 and L2 be non-vertical lines with slopes m1 and m2, respectively

(a) The lines are parallel if and only if

$$m_1 = m_2 \quad \ldots\ldots \quad (3)$$

(b) The lines are perpendicular if and only if

$$m_1 m_2 = -1 \quad \ldots\ldots\ldots \quad (4)$$

Basically, if two lines are parallel, then they have the same slope, and if they are perpendicular, then the product of there slopes is -1.

Formula (4) can be rewritten in the form

$$m_2 = -\frac{1}{m_1}$$

In words, this tells us that two non-vertical lines are perpendicular if and only if their slopes are NEGATIVE RECIPROCALS OF ONE ANOTHER

Example

Use slopes to show that the points A(I, 3), B(3, 7), and C(7, 5) are vertices of a right triangle.

Solution:

Slope through A and B = m1 = (7-3)/(3-1) = 2

Slope through B and C = m2 = (5-7)/(7-3) = - 1/2

Since m1m2 = (2)(-1/2)=-1

The line through A and B is perpendicular to the line through B and C; thus, ABC is a right triangle.

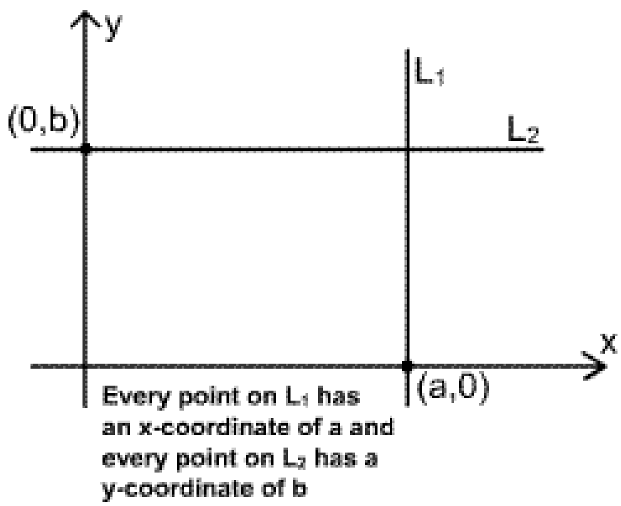
Equations of Lines

Lines Parallel to the Coordinate axes

We now turn to the problem of finding equations of lines that satisfy specified conditions.

The simplest cases are lines PARALLEL TO THE COORDINATE AXES: A line parallel to the *y*-axis intersects the *x*-axis at some point (a, 0).

This line consists precisely of those points whose *x*-coordinate is equal to a. Similarly, a line parallel to the x-axis intersects the y-axis at some point (0, b). This line consists precisely of those points whose y-coordinate is equal to b.



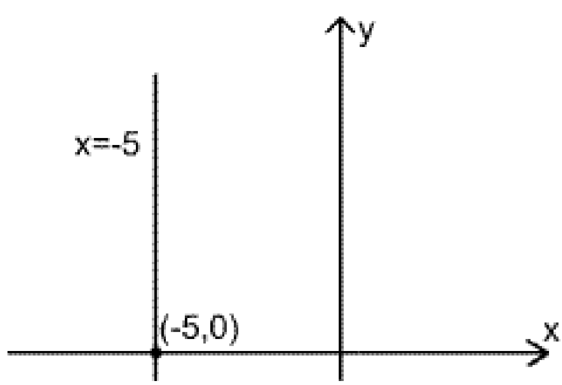
Theorem 1.4.5

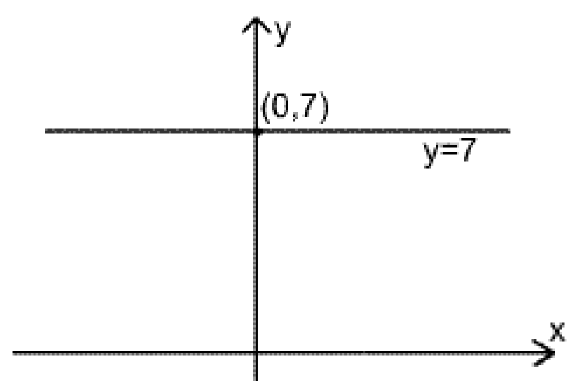
The vertical line through (a,0) and the horizontal line through (0,b) are represented, respectively, by the equations

$$x = a \quad \text{and} \quad y = b$$

Example

The graph of x = -5 is the vertical line through (-5,0) and the graph of y = 7 is the horizontal line through (0,7).





LINES DETERMINED BY POINT AND SLOPE

There are infinitely many lines that pass through any given point in the plane. However, if we specify the slope of the line in addition to a point on it, then the point and the slope together determine a unique line.

Let us see how we can find the equation of a non-vertical line L that passes through a point P1(x1,y1) and has slope m. If P(x,y) is any point on L, different from P1, then the slope m can be obtained from the points P(x,y) and P1(x1,y1); this gives

$$m = \frac{y - y_1}{x - x_1}$$

which gives

$$y - y_1 = m(x - x_1)$$

In summary, we have the following theorem.

Theorem 1.4.6

The line passing through P1(x1,y1) and having slope m is given by the equation

$$y - y_1 = m(x - x_1)$$

This is called the **point-slope** form of line.

Example

Write an equation for the line through the point (2,3) with slope

-3/2.

Solution:

We substitute x1=2, y1=3 and m=-3/2 into the point-slope equation and obtain

y-3 = -3/2 (x-2) on simplification y = -3/2 x + 6

Example

Write an equation for the line through the point (-2,-1) and (3,4).

Solution: The line's slope is

m = (-1-4) / (-2-3) = -5 / -5 = 1

We can use this slope with either of the two given points in the point-slope equation

With $(x1,y1) = (-2,-1)$

$y = -1 + 1(x-(-2))$

$y = -1+x+2$

$y = x+1$

With $(x1,y1) = (3,4)$

$y = 4 + 1(x-3)$

$y = 4 + x - 3$

$y = x+1$

Lines Determined by Slope and y-Intercept

A nonvertical line crosses the y-axis at some point (0,b). If we use this point in the point slope form of its equation, we obtain

$$y-b = m(x-0)$$

Which we can rewrite as $y = mx+b$

Theorem 1.4.7

The line with y-intercept b and slope m is given by the equation

$$y = mx+b$$

This is called the **slope-intercept form** of the line.

Example

The line $y = 2x-5$ has slope 2 and y-intercept -5.

Example

Find the slope-intercept form of the equation of line with slope -9 and that crosses the y-axis at (0,-4)

Solution : We are given with m=-9 and b=-4, so slope-intercept form of line is

$$y = -9x-4$$

Example

Find slope-intercept form of the equation of line that passes through (3,4) and (-2,-1).

Solution :

The line's slope is

$$m = (-1-4) / (-2-3) = -5 / -5 = 1$$

We can use this slope with either of the two given points in the point-slope equation

With $(x1,y1) = (-2,-1)$

$$y - (-1) = 1\,(x - (-2))$$

$$y + 1 = x + 2$$

$$y = x + 1$$

Which is required slope-intercept form.

The General Equation of a Line

An equation that is expressible in the form

$$Ax + By + C = 0$$

Where A, B and C are constants and A and B are not both zero, is called a first-degree equation in x and y. For example

$$4x + 6y - 5 = 0$$

is a first-degree equation in x and y.

Theorem 1.4.8

Every first degree equation in x and y has a straight line as its graph and, conversely, every straight line can be represented by a first-degree equation in x and y.

Example

Find the slope and y-intercept of the line $8x + 5y = 20$

Solution :

Solve the equation for y to put it in the slope-intercept form, then read the slope and y-intercept from equation

$$5y = -8x + 20$$

$$y = -8/5\,x + 4$$

The slope is $m = -8/5$ and the y-intercept is $b = 4$

Applications

The Importance of Lines and Slopes

Light travel along with lines, as do bodies falling from rest in a planet's gravitational field or coasting under their own momentum (like hockey puck gliding across the ice). We often use the equations of lines (called **Linear equations**) to study such motions

Many important quantities are related by linear equations. One we know that a relationship between two variables is linear, we can find it from the any two pairs of corresponding values just as we find the equation of a line from the coordinates of two points.

Slope is important because it gives us way to say how steep something is (roadbeds, roofs, stairs). The notion of slope also enables us to describe how rapidly things are changing. For this reason it will play an important role in calculus.

Example

Fahrenheit temperature (F) and Celsius temperature (F) are related by a linear equation of the form

$$F = mC + b$$

The freezing point of water is $F = 32$ or $C = 0$,

while the boiling point is

$$F = 212 \quad \text{or} \quad C = 100 \; .$$

Thus

$$32 = 0m + b \;\; \text{and} \;\; 212 = 100m + b$$

Solving these two equation we get

$$b = 32 \quad \text{and} \quad m = 9/5 \;, \text{therefore}$$

$$F = 9/5\, C + 32$$