Lecture # 35

Volume by Cylindrical Shells

- Cylindrical Shells
- Volume of a Cylindrical Shell
- Cylindrical Shells centered on the y-axis
- Volume through the surface area of a surface created from a cylindrical shell

Cylindrical Shells

If we take a washer – a disk with a hole in it – and extend it UP, we generate a solid called a cylindrical shell.

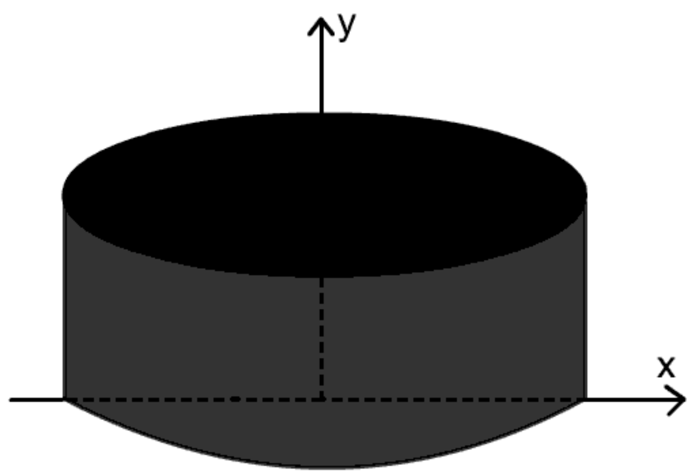
This is a solid confined by two concentric right circular cylinders

The Volume of a cylindrical shell can be expressed as

$$V = (\text{area of cross section}).(\text{hieght}) = (\pi\ r_2^{\ 2} - \pi\ r_1^{\ 2})h$$

$$= \pi\ (r_2 + r_1)(r_2 - r_1)h$$

$$= 2\pi \left[\frac{1}{2}(r_2 + r_1)\right].h.(r_2 - r_1)$$



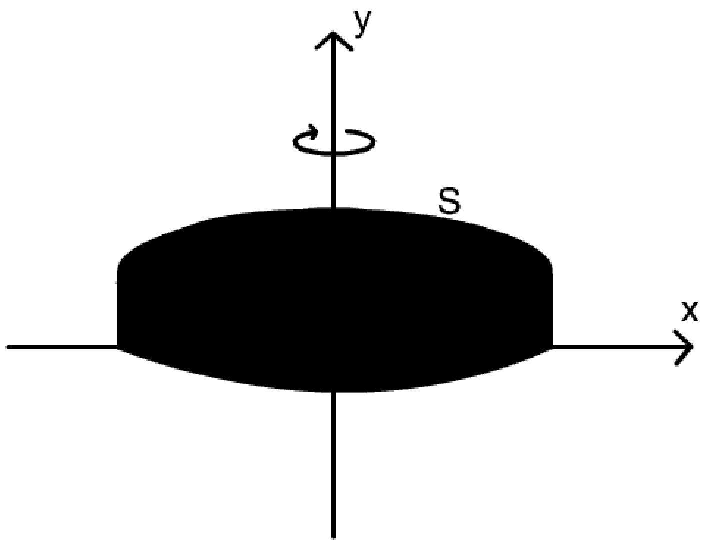
Let's rewrite this rearrangement of the Volume of a cylindrical shell as

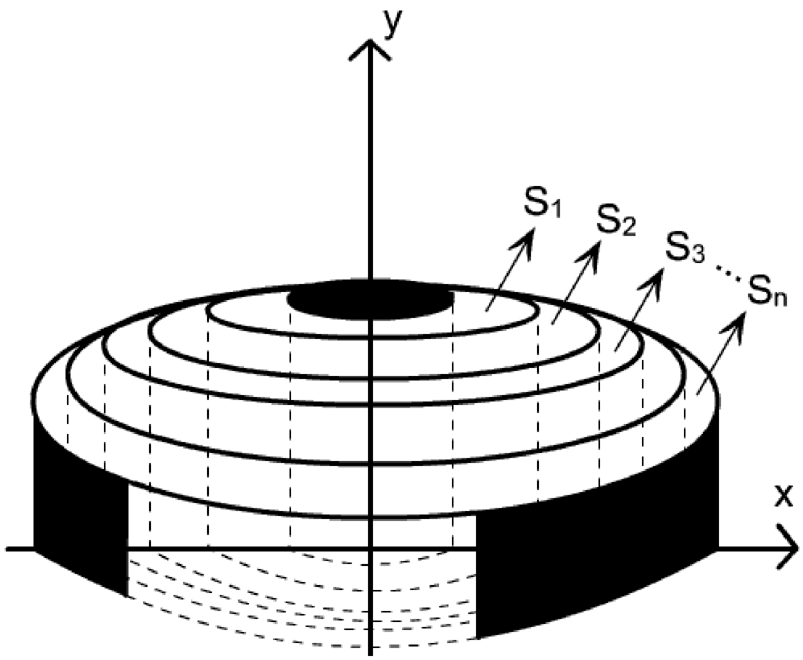
V = 2π (average radius) .(height) .(thickness)

Now we can use this formula to compute the Volume of a Solid generated by revolution of a surface around an axis.

Consider the following

R is a region bounded by the graph of f (x) on top, below by the x-axis, and to the left by x = a, and right by x = b





When we revolve R around the x-axis , we get the solid S as in the picture.

To find the volume of S, here is what we do.

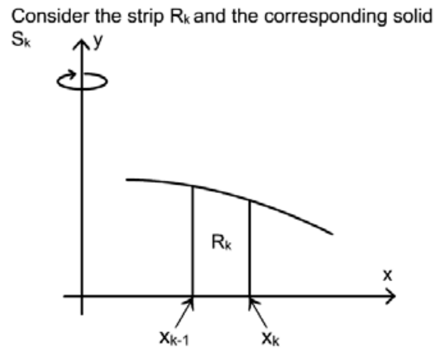
Subdivide [a,b] into *n* subintervals with widths $\Delta x_1 ..., \Delta x_n$ by inserting the points $x_1 ..., x_n$ between a and b.

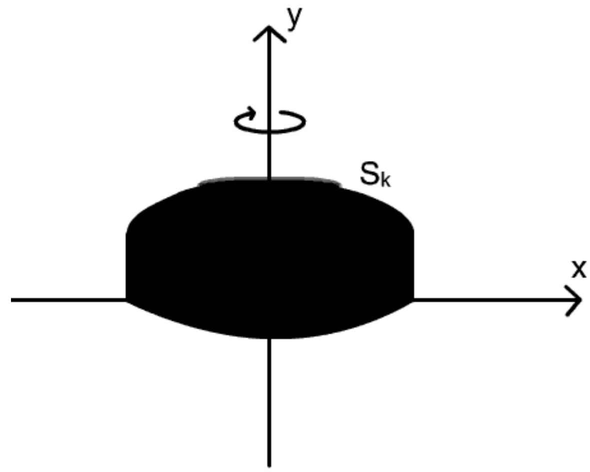
If we draw vertical lines from these points to the graph of *f*, we get subdivisions $R_1, ..., R_n$ of the region R.

Revolving these strips around the y-axis gives solids $S_1, ..., S_n$

So we can do the following to get the volume of S

$$V(S) = V(S_1)+...+V(S_n)$$





Consider the strip R_k and the corresponding solid S_k.

The solid S_k is not necessarily a cylindrical shell as it may have a curve upper surface depending on the graph of f.

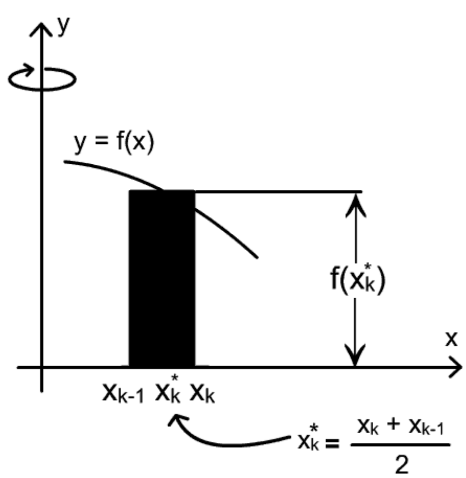
However, if the width of Δx_k is very small, then we can get a good approx to the region R_k by the rectangle of width Δx_k and height

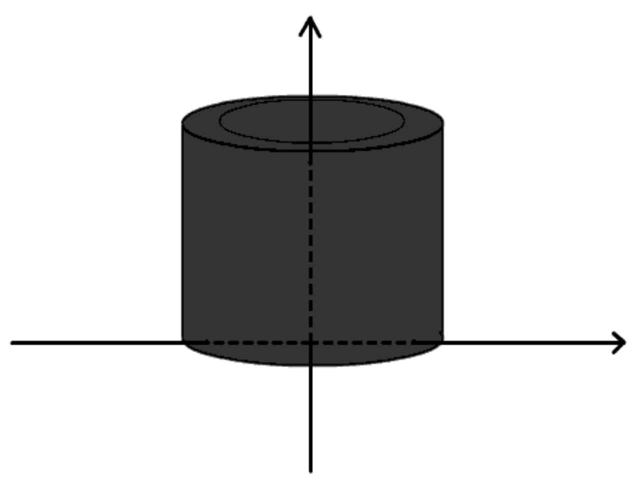
$f(x_k^*)$ where

$$x_k^* = \frac{x_k + x_{k+1}}{2}$$

Is the midpoint of the interval $\left[x_{k-1}^*, x_k^*\right]$

This in turn will give us a cylindrical shell when we revolve it around y-axis. which will be a good approx to the solid.





The volume of S_k can now be approximated using the width height f(x_k^*), average radius x_k^* of the cylindrical shell as Δx_k

$$V(S_k) \approx \text{volume of approximating cylinderical shell } = 2\pi\ x_k^* f(x_k^*)\Delta x_k$$

The volume of the whole solid S will now be just the sum

$$V(S) = \sum_{k=1}^{n} 2\pi\ x_k^{\ *} f(x_k^{\ *})\Delta x_k$$

We can take the limit and get the exact Volume of S as

$$V(S) = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} 2\pi\ x_k^{\ *} f(x_k^{\ *})\Delta x_k = \int_a^b 2\pi\ x f(x)dx$$

In this process, we see that we have a formula to computed volume of a solid which is got from revolving a region around the Y-AXIS

Cylindrical Shells centered on the y-axis

VOLUME FORMULA

Let R be a plane region bounded above by a continuous curve y = f(x) below by x-axis, and on the left and right , respectively, by the lines x = a and x = b. Then the volume of the solid generated by revolving R about the y-axis is given by

$$V = \int_a^b 2\pi x f(x) dx$$

Example 1

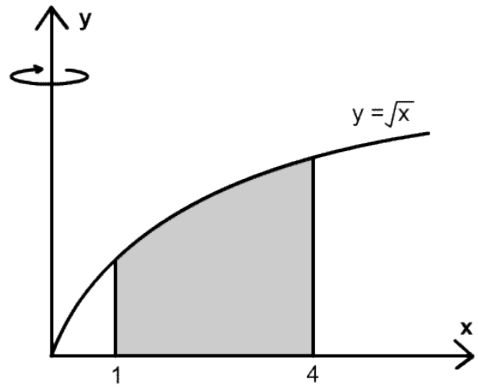
Use cylindrical shells to find the volume of the solid generated by the region enclosed between $y = \sqrt{x}, x = 1, x = 4$ and the x-axis is revolved about the y-axis.

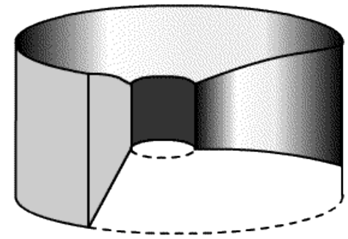
Solution:

Since $f(x) = \sqrt{x}, a = 1, and\ b = 4$ and the above volume formula yields

$$V = \int_1^4 2\pi x\sqrt{x}dx = 2\pi\int_1^4 x^{\frac{3}{2}}dx$$

$$V = 2\pi\left[\frac{2}{5}x^{\frac{5}{2}}\right]_1^4 = \frac{124\pi}{5}$$





Cutaway view of the solid

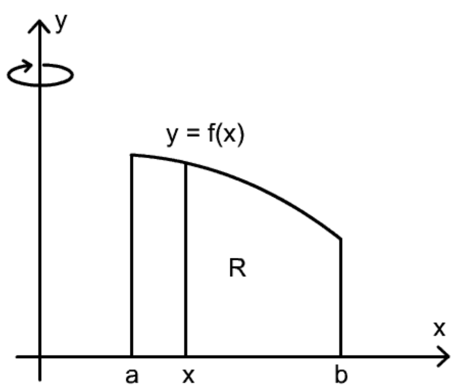
Volume through the surface area of a surface created from a cylindrical shell

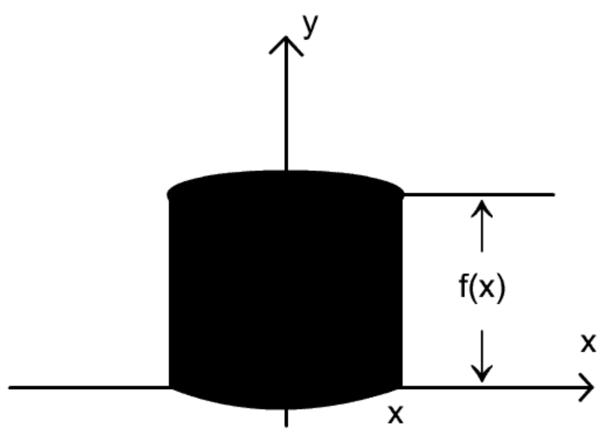
A bit more general and perhaps easier to conceptualize method can be worked out for cylindrical shells

Instead of thinking of the solid S_k as having a width Δx_k suppose that it has no width

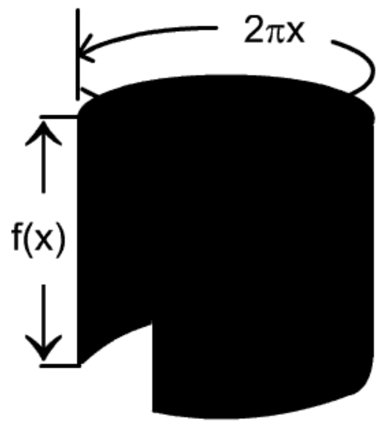
This would mean that S_k, will be a straight forward right circular cylinder having radius x, height f (x).

The surface area of this cylinder will be $2\pi\ xf(x)$.





The surface area of this cylinder will be 2π x f(x)



The surface area of this cylinder will be

$$2\pi\ x\ f(x)$$

This is exactly the integrand in the Volume formula we saw earlier. So this really means that :

VOLUME **V** BY CYLIDRICAL SHELLS IS THE INTEGRAL OF THE SURFACE AREA GENERATED BY AN ARBITRARY SECTION OF THE REGION **R** TAKEN PARALLEL TO THE AXIS ABOUT WHICH R IS REVOLVED.

This view of volume by cylindrical shells helps us do calculation in more general setting where for example the lower boundary may not be interval

Example 2

Use cylindrical shells to find the volume of the solid when the region R in the first quadrant enclosed between $y = x, and\ y = x^2$

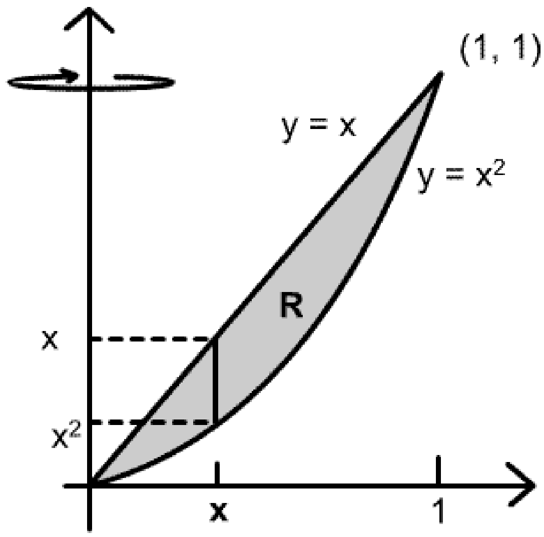
Is revolved about the y-axis

Solution:

At each x in [0,1] the cross section of R parallel to the y-axis generates a cylindrical surface of height $x - x^2$ and radius x. Since the area of the surface is $2\pi x(x - x^2)$.Thus the volume of the solid is given by the following formula and the figure is also below

$$V = \int_0^1 2\pi x(x - x^2)dx = 2\pi \int_0^1 (x^2 - x^3)dx$$

$$V = 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{\pi}{6}$$


(1, 1)
y = x
y = x2
R
x
x2
x
1