MTH101 Solution: Practice Questions
Lecture No. 35 to 37

Question 1: Use cylindrical shells to find the volume of the solid generated when the region 'R' in the first quadrant enclosed between $y = 4x$ and $y = x^3$ is revolved about the y-axis.

Solution:

To find the limit of integration, we will find the point of intersection between two curves.
Equating $y = 4x$ and $y = x^3$, we get

$$x^3 = 4x,$$
$$\Rightarrow x^3 - 4x = 0,$$
$$\Rightarrow x(x^2 - 4) = 0,$$
$$\Rightarrow x = 0, x = \pm 2.$$

Since our region R is in first quadrant so we ignore $x = -2$. Hence limit of integration is $x = 0$ to $x = 2$.

So $V = \int_0^2 2\pi x(4x - x^3)dx,$

$$= 2\pi \int_0^2 (4x^2 - x^4)dx,$$

$$= 2\pi \left| 4\frac{x^3}{3} - \frac{x^5}{5} \right|_0^2.$$

After simplification, we get

$$V = \frac{128}{15}\pi.$$

Question 2: Use cylindrical shell method to find the volume of the solid generated when the region enclosed between $y = x^3$ and the x-axis in the interval $[0,3]$ is revolved about the y-axis.

Solution:

$$V = \int_0^3 2\pi\, x(x^3)\, dx = 2\pi \int_0^3 x^4\, dx,$$

$$\left(\begin{array}{l} \because \text{Here } f(x) = x^3, \quad a = 0; \ b = 3. \\ \text{Cylindrical shells revolved about the } y\text{-axis}: V = \int_a^b 2\pi\, x\, f(x)\, dx \end{array} \right)$$

$$= 2\pi . \frac{1}{5} \left| x^5 \right|_0^3.$$

After simplification, we get

$$V = \frac{486}{5}\pi.$$

Question 3: Find the arc length of the curve $y=\frac{2}{3}(x-1)^{\frac{3}{2}}$ from $x=0$ to $x=\frac{1}{2}$.

Solution:

$$\because \quad y=\frac{2}{3}(x-1)^{\frac{3}{2}},$$

$$\Rightarrow \frac{dy}{dx}=\frac{2}{3}\left(\frac{3}{2}\right)(x-1)^{\frac{1}{2}}=(x-1)^{\frac{1}{2}}.$$

$$\because \quad L=\int_a^b \sqrt{1+[f'(x)]^2}\,dx,$$

$$\Rightarrow L=\int_0^{\frac{1}{2}} \sqrt{1+\left[(x-1)^{\frac{1}{2}}\right]^2}\,dx=\int_0^{\frac{1}{2}} \sqrt{(1+x-1)}\,dx=\int_0^{\frac{1}{2}} \sqrt{x}\,dx,$$

$$=\frac{2}{3}x^{\frac{3}{2}}\Big|_0^{\frac{1}{2}},$$

$$=\frac{2}{3}\left(\frac{1}{2}\right)^{\frac{3}{2}},$$

$$=\frac{1}{3\sqrt{2}}.$$

Question 4: If f is a smooth function on $[1,4]$, then find a definite integral indicating the arc length of the curve $x=y^{\frac{2}{3}}$ from $y=1$ to $y=4$. **Note**: Do not evaluate it further.

Solution:

Here $\quad x=g(y)=y^{\frac{2}{3}},$

$$\Rightarrow \quad g'(y)=\frac{2}{3}y^{-\frac{1}{3}}.$$

$$\because \text{Arc length} \quad L=\int_a^b \sqrt{1+[g'(y)]^2}\;dy,$$

$$\therefore \quad L=\int_1^4 \sqrt{1+\frac{4}{9}y^{\frac{-2}{3}}}\;dy.$$

Question 5: Find the area of the surface generated by revolving the curve $y=\sqrt{1-x^2}$; $0\le x\le 1$ about the x – axis.

Solution:

$$\because\ f(x) = y = \sqrt{1-x^2},$$

$$\Rightarrow\ f'(x) = -\frac{x}{\sqrt{1-x^2}}.$$

$$\because\ \ S = \int_a^b 2\pi f(x)\sqrt{1+\left[f'(x)\right]^2}\,dx,$$

$$\Rightarrow S = \int_0^1 2\pi\sqrt{1-x^2}\sqrt{1+\left[-\frac{x}{\sqrt{1-x^2}}\right]^2}\,dx,$$

$$= \int_0^1 2\pi\sqrt{1-x^2}\sqrt{1+\frac{x^2}{1-x^2}}\,dx,$$

$$= \int_0^1 2\pi\sqrt{1-x^2}\sqrt{\frac{1-x^2+x^2}{1-x^2}}\,dx,$$

$$= \int_0^1 2\pi\,dx,$$

$$= 2\pi x\Big|_0^1,$$

$$= 2\pi.$$

Question 6: Write a definite integral indicating the area of the surface generated by revolving the curve $y = x^2$; $0 \le x \le 2$ about the x – axis. **Note**: Do not evaluate it further.

Solution:

$$\because\quad y = f(x) = x^2\ \ ;\ \ 0 \le x \le 2,$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = 2x.$$

$$\therefore\ \ S = \int_0^2 2\pi\left(x^2\right)\sqrt{1+(2x)^2}\ dx, \qquad \left(\because S = \int_c^d 2\pi\ f(x)\sqrt{1+\left[f'(x)\right]^2}\ dx\right)$$

$$= 2\pi\int_0^2 x^2\sqrt{1+4x^2}\ dx.$$