Lecture # 34

Volume by Slicing; Disks and Washers

- Definite Integrals to find Volumes of three dimensional solids
- Cylinders
- The method of Slicing
- Volumes by cross sections perpendicular to the x axis
- Volumes by cross sections perpendicular to the y axis
- Volumes of solids of revolution by:
    i. Volumes by Disks perpendicular to x axis
    ii. Volumes by Disks perpendicular to y axis
    iii. Volumes by washers perpendicular to x axis
    iv. Volumes by washers perpendicular to y axis

Cylinders

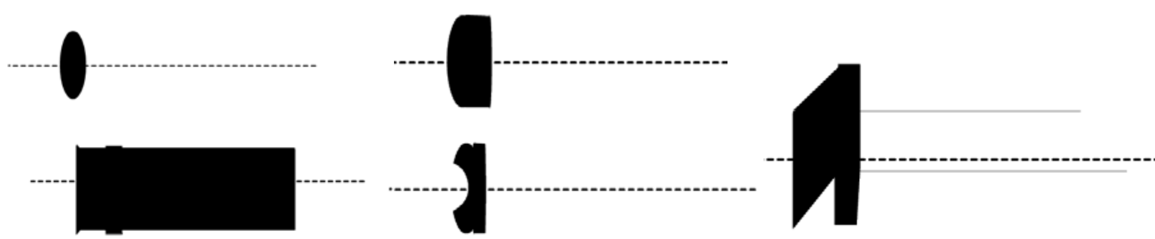
How to go from 0-d to 3-d and beyond in terms of a cube.

We can get other 3d solids by moving 2d planes in a certain fashion in 3d space.

If I move a 2d circle along a line that is perpendicular to the circle, I will get a cylinder.

If I move a circle with a hole in it, or a washer, along a line perpendicular to the washer, then I will get a cylinder with a hole in it.

In general, if I move a 2d plane in a direction along a line perpendicular to the region, then I get a RIGHT CYLINDER.

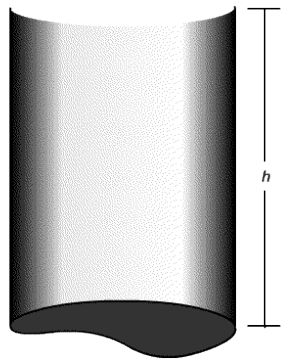


Every solid object has Volume, Volume is the 3d area!

We can find the volume of a right cylinder generated by moving a 2d plane by the following formula

$$V = A.h$$

Where A is the area of the 2d object, and h is the height of the cylinder



Volume = A. *h*

So the Volume of a cylinder is the area of a cross section of the cylinder multiplied by the height of the cylinder.

But there are 3d solids that are not right cylinders and are neither made up of finitely many right cylinders.

So we cannot use the formulas for the area of the right cylinders to find the area of such an object.

In this case we use the technique of slicing.

The method of Slicing

Here is such a 3d solid

This is not made up of finitely many right cylinders.

We find the volume of this object by slicing.

Impose an x axis on the solid.

Then we can imagine that the solid is bounded to the left by a plane perpendicular to the x- axis at x = a and to the right by a plane at x = b

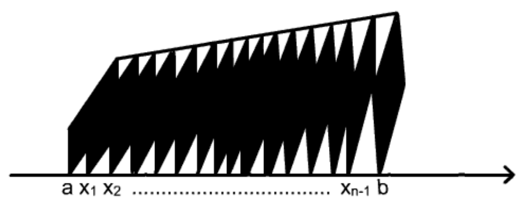
The problem with finding the volume of this non-cylindrical solid is clear now: at any point along the x-axis, the cross sections perpendicular to the x-axis will have different areas!

Let's call the area of a cross section at some arbitrary point x A (x)

Now Let's divide the interval [a,b] into *n* subintervals of width $\Delta x_1, \Delta x_2, ..., \Delta x_n$ By inserting the points $x_1, x_2, ..., x_n$ between a and b.

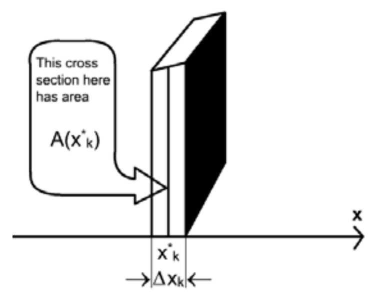
We can now pass a plane perpendicular to the x-axis through each of these points.

These planes will subdivide the solid into SLICES $S_1, S_2, ..., S_n$.



Consider a typical slice S_k as shown in the figure

If this slice is very thin, then the cross section of this slice S_k will not vary too much in the interval Δx_k which defines the thickness of the slice S_k. That is, the cross section will have the same shape in the interval if Δx_k is thin.



So let's choose a point x_k^* in the interval Δx_k and take a cross section of the slice at this point. Then the area of this cross section will be A(Δx_k)

The volume of the slice S_k then will be approximately

$$V_k \approx A(x_k^*)\Delta x_k$$

We can use this formula for all the THIN slices we make to get their respective volumes.

The total volume of the solid can be approximated by

$$V = V_1 + ... + V_n \approx \sum_{i=1}^{n} A(x_k^*)\Delta x_k$$

Now increase the number of slices. Then the slices will become VERY THIN and the approx will get better and better. So we can take the limit to get

$$\max \Delta x_k \to 0$$

$$V = \lim_{\max \Delta x_k \to 0} \sum_{i=1}^{n} A(x_k^{*})\Delta x_k = \int_a^b A(x)dx$$

Volumes by cross sections perpendicular to the x axis

VOLUME FORMULA

Let S be a solid bounded by two parallel planes perpendicular to the x-axis at x = a and x = b. If , for each x in [a,b], the cross-sectional area of S perpendicular to the x-axis is A(x), then the volume of the solid is

$$V = \int_a^b A(x)dx$$

Volumes by cross sections perpendicular to the y axis

VOLUME FORMULA

Let S be a solid bounded by two parallel planes perpendicular to the y-axis at y = c and y = d. If , for each y in [c,d], the cross-sectional area of S perpendicular to the y-axis is A(y), then the volume of the solid is

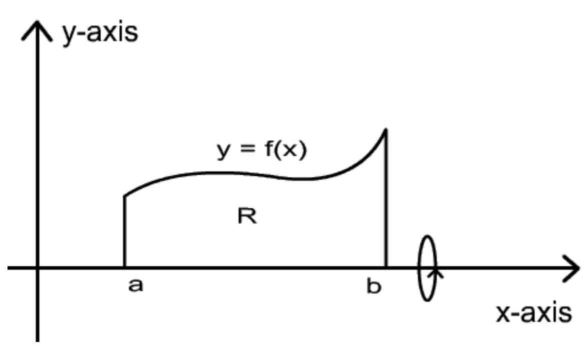
$$V = \int_c^d A(y)dy$$

Volumes of solids of revolution by:

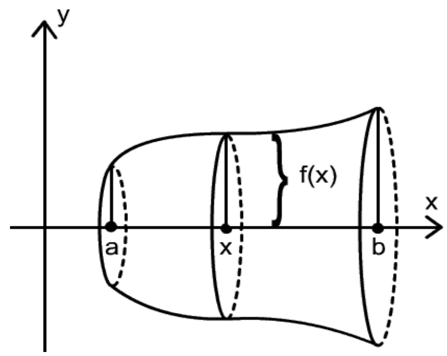
I. Disks perpendicular to x-axis

Let *f* be a nonnegative and continuous function on [a,b]

Let R be the region bounded above by the graph of *f* and on the sides by lines x = a and x = b



If this solid is revolved around the x-axis, it generates a solid with a circular cross sections.



The cross section at x has radius f(x) so the area of this cross section is

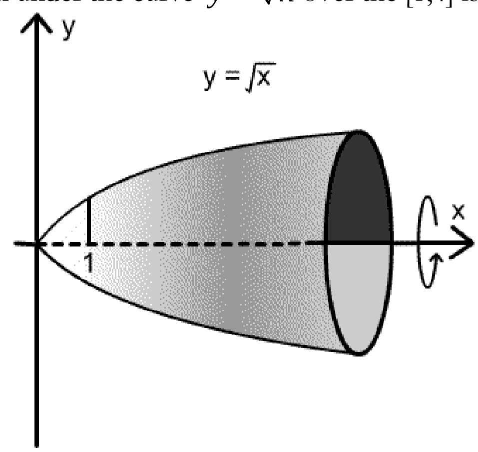
$$A(x) = \pi\left[f(x)\right]^2$$

So from the formulas we just saw we can write for the volume of this solid:

$$V = \int_a^b \pi\left[f(x)\right]^2 dx$$

Example 2

Find the volume of solid that is obtained when the region under the curve $y = \sqrt{x}$ over the [1,4] is revolved about the x-axis .



Solution:

From $V = \int_a^b \pi\left[f(x)\right]^2 dx$,the volume is

$$V = \int_a^b \pi\left[f(x)\right]^2 dx = \int_1^4 \pi x dx$$

on evaluating

$$V = \frac{15\pi}{2}$$

Example 3

Derive the formula for the volume of a sphere of radius r.

Solution:

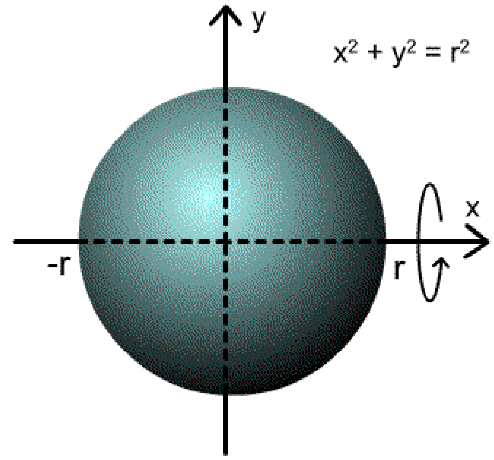
As indicated in the figure , a sphere of radius r can be generated by revolving the upper half of the circle

$$x^2 + y^2 = r^2$$

About the x-axis. Since upper half of this circle is graph of

$$y = f(x) = \sqrt{r^2 - x^2}$$

it follows from above formula that the volume of the sphere is



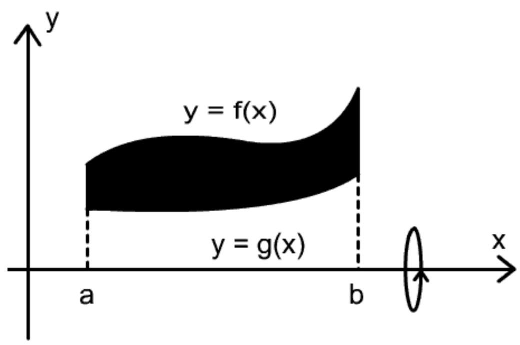
$$V = \int_a^b \left[f(x) \right]^2 dx = \int_{-r}^{r} \pi \left(r^2 - x^2 \right) dx$$

On evaluating

$$V = \frac{4}{3}\pi r^3$$

Volumes by washers perpendicular to x axis

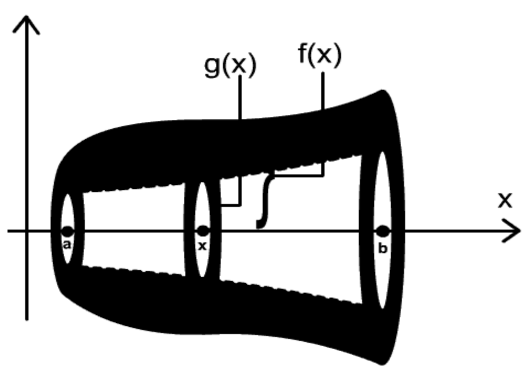
Now consider this picture



Here R is a region confined by the graph of two non-negative continuous functions *f* and *g* with f greater than or equal to g on the interval [a,b]

When this region is revolved around the x-axis, we get a solid with a hole in the center

This solid has a cross section that looks like a washer.



This washer cross section has two radii defined by the function f and g as is in the picture.

So the area of this cross section will be

$$A(x) = \pi \left[f(x) \right]^2 - \pi \left[g(x) \right]^2 = \pi \left(\left[f(x) \right]^2 - \left[g(x) \right]^2 \right)$$

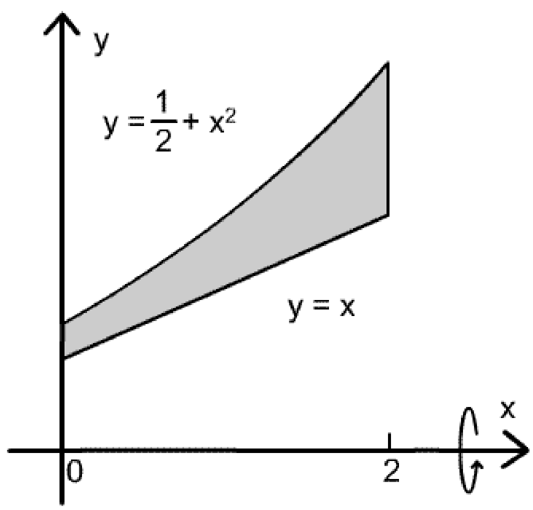
And the volume of the solid now can be expressed

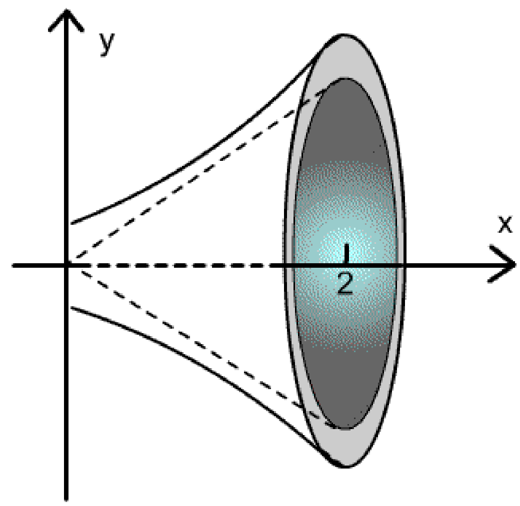
$$V = \int_a^b \pi \left(\left[f(x) \right]^2 - \left[g(x) \right]^2 \right)$$

Example 4

Find the volume of the solid generated when the region between the graph of

$f(x) = \frac{1}{2} + x^2$ *and* $g(x) = x$ over the interval [0,2] Is revolved about the x-axis .





Solution;

From

$$V = \int_a^b \pi \{ \left[f(x) \right]^2 - \left[g(x) \right]^2 \} dx$$

the volume is

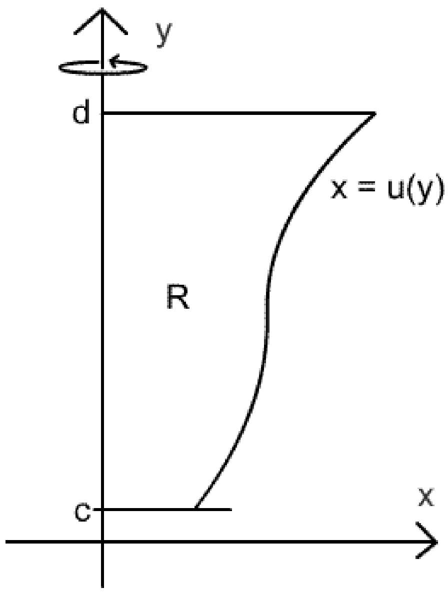
$$V = \int_a^b \pi \{ \left[f(x) \right]^2 - \left[g(x) \right]^2 \} dx = \int_0^2 \pi \left(\left[\frac{1}{2} + x^2 \right] - x^2 \right) dx$$

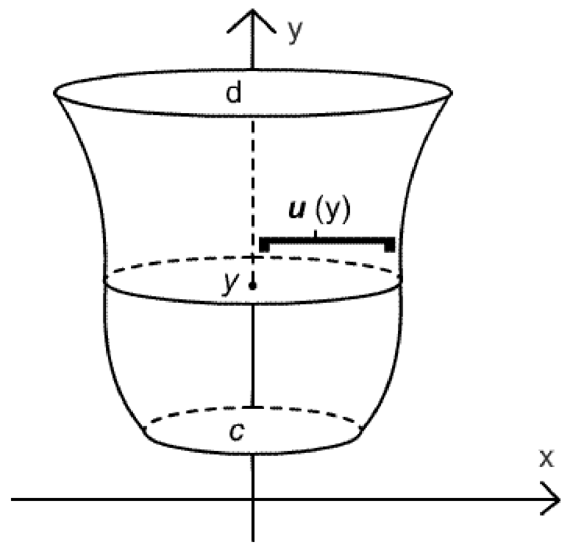
$$= \int_0^2 \pi \left(\frac{1}{4} + x^4 \right) dx = \frac{69\pi}{10} \qquad (\text{on evaluating})$$

Volume by Disks perpendicular to y axis

Just as we found the volumes of solids generated by revolving a region around the x-axis, we can find the volume of solids generated by revolving a region around the y-axis in which the cross section is a disk.

Here is how in a picture



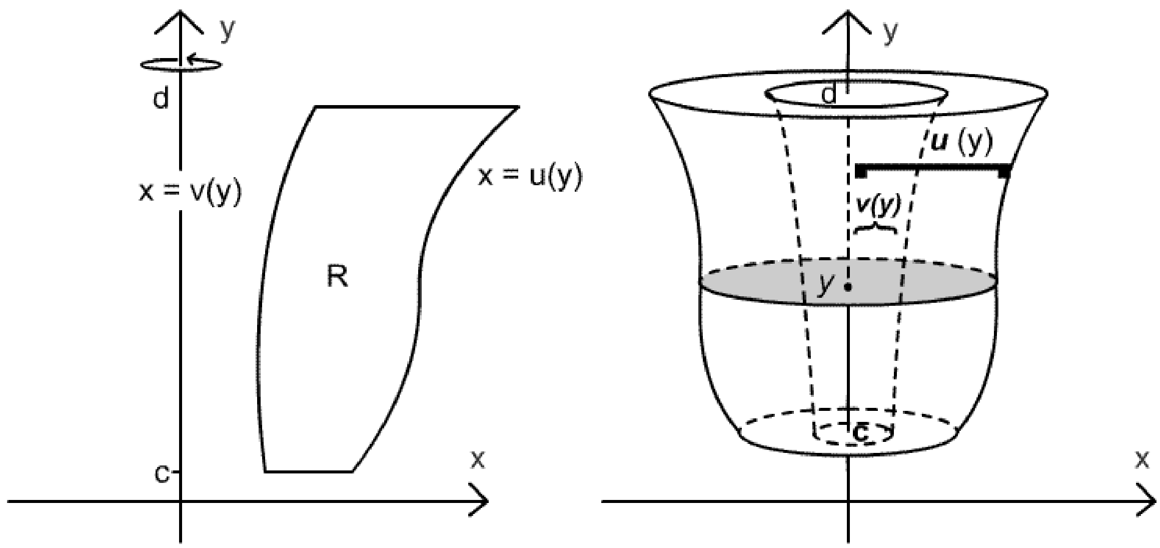


The formula in this case for volume is the following

$$V = \int_c^d \pi \left[u(y)\right]^2 dy$$

Volume by Washers perpendicular to y axis

Just as we found the volumes of solids generated by revolving a region around the x-axis, we can find the volume of solids generated by revolving a region around the y-axis in which the cross section is a washer



The formula expressing it is $V = \int_c^d \pi \left(\left[u(y)\right]^2 - \left[v(y)\right]^2 \right) dy$